JOURNAL OF FUNCTIONAL ANALYSIS **146**, 236–298 (1997)
ARTICLE NO. FU963031

On the Attracting Orbit of a Nonlinear Transformation Arising from Renormalization of Hierarchically Interacting Diffusions

Part II: The Non-compact Case

J.-B. Baillon

Institut de Mathématiques et Informatique, Université de Lyon I, 43 Boulevard du 11 Novembre 1918, F-69622 Villeurbanne Cedex, France

Ph. Clément

Faculteit der Technische Wiskunde en Informatica, Technische Universiteit Delft, Mekelweg 4, NL-2600GA Delft, The Netherlands

A. Greven

Mathematisches Institut, Universität Erlangen–Nürnberg, Bismarckstrasse 1$\frac{1}{2}$, D-91054 Erlangen, Germany

and

F. den Hollander

Mathematisch Instituut, Universiteit Nijmegen, Toernooiveld 1, NL-6525ED Nijmegen, The Netherlands

Received July 1996; accepted July 1996

This paper analyzes the n-fold composition of a non-linear integral operator acting on a class of functions on $[0, \infty)$. Attracting orbits and attracting fixed points are identified. Various results of convergence to these orbits and to these fixed points are derived. The proofs are based on order-preserving properties and comparison techniques. A key role is played by the eigenfunctions of the operator, which are used as comparison objects. The results imply that the space-time scaling limit of an infinite system of interacting diffusions has universal behavior independent of model parameters. The paper can be read independently of Part I.

0022-1236/97 $25.00

CONTENTS

0. INTRODUCTION AND MAIN RESULTS

The present paper studies the iterates of a non-linear transformation F acting on a class of functions g: $[0, \infty) \to [0, \infty)$. The problem arises in a probabilistic context, namely, the study of systems of hierarchically interacting diffusions ([CGS], [DG1-4], [DGV]). This study is part of a larger area, where the goal is to understand universal behavior on large space-time scales of stochastic systems with interacting components (see e.g., [dMP], [Sp]). Part II, like Part I, focusses on the analytic aspects of the transformation (and can be read independently of Part I). Some probabilistic motivation is given in Part I Sections 0.3 and 0.6 and Part II Section 0.6.

The transformation F that will be studied in this paper plays the role of a *renormalization transformation* for an infinite system of diffusions, taking values in $[0, \infty)$ and interacting with each other in a hierarchical fashion. The iterates $F^n g$ $(n \geqslant 0)$ describe the behavior of this system along an *infinite hierarchy of space-time scales* indexed by n. The transformation F acts on a function g: $[0, \infty) \to [0, \infty)$ (chosen from an appropriate class) playing the role of the local diffusion rate for the single components in the system. The nth iterate $F^n g$ is the local diffusion rate of a typical block average on space-time scale n (see Section 0.6 below).

In Part I we considered a similar system, but with the diffusions taking values in $[0, 1]$. Therefore we had a similar transformation, but acting on a different class of functions. In Part II we continue our study for functions on $[0, \infty)$. We again analyze the orbit of the transformation. However, the behavior of the orbit is qualitatively *very* different from that in Part I and therefore new techniques are required. The non-compactness of $[0, \infty)$ gives rise to new phenomena.

Renormalization transformations come up in many other interacting stochastic systems. Sometimes a rigorous implementation is possible (see e.g., [BM], [D], [BK]), but frequently the transformation is so untractable that one has to resort to non-rigorous approximation techniques (see [DoG] for an overview in a statistical mechanics context). In some systems the transformation is even at risk of not being properly definable (see [EFS]).

0.1. *The Transformation*

Let $\mathscr{H}$ be the class of functions g: $[0, \infty) \to [0, \infty)$ satisfying

$$\begin{aligned} &\text{(i)} \quad g(0)=0 \\ &\text{(ii)} \quad g(x)>0 \qquad \text{for} \quad x>0 \\ &\text{(iii)} \quad g \text{ locally Lipschitz continuous on } [0, \infty) \\ &\text{(iv)} \quad \lim_{x \to \infty} x^{-2} g(x)=0. \end{aligned} \tag{0.1}$$

For $g \in \mathscr{H}$, let $(\nu_\theta^g)_{\theta \in [0, \infty)}$ be the family of probability measures on $[0, \infty)$ given by $\nu_0^g = \delta_0$ (the point measure at 0) and

$$\nu_\theta^g(dx) = \frac{1}{Z_\theta^g} \left\{ \frac{1}{g(x)} \exp\left[-\int_\theta^x \frac{y-\theta}{g(y)}\, dy \right] \right\} dx \qquad (\theta > 0), \tag{0.2}$$

where Z_θ^g is the normalizing constant. Define the transformation F acting on $g \in \mathscr{H}$ as

$$(Fg)(\theta) = \int_0^\infty g(x)\, \nu_\theta^g(dx) \qquad (\theta \in [0, \infty)). \tag{0.3}$$

Since $\nu_\theta^g(dx)$ itself depends on g the transformation F is *non-linear*. The starting point of our analysis is the following:

Lemma 1. (i) *F is well defined on $\mathscr{H}$.*

(ii) *$F\mathscr{H} \subset \mathscr{H}$.*

(iii) *For all $g \in \mathscr{H}$ the function $\theta \to (Fg)(\theta)$ is C^∞ on $(0, \infty)$.*

In Section 0.6 we shall explain how (0.1–0.3) arise naturally from a space-time scaling analysis of a system of interacting diffusions on $[0, \infty)$. We shall see that ν_θ^g is the *equilibrium* measure of a single diffusion with drift towards θ and with local diffusion rate given by g (see (0.26) below). Hence $(Fg)(\theta)$ is the *average diffusion rate in equilibrium* as a function of the drift parameter θ (see (0.30) below).

0.2. *The Orbit*

Our main object of study will be the *orbit*

$$\{F^n g\}_{n=0}^{\infty}. \tag{0.4}$$

Our goal will be to identify *subclasses* of $\mathscr{H}$ for which there exists a $g^* \in \mathscr{H}$ and a sequence $(d_n)_{n \geqslant 0} \subset (0, \infty)$ such that

$$\lim_{n \to \infty} d_n F^n g = g^*, \tag{0.5}$$

either pointwise or in a suitable norm, satisfying

(i) g^* and d_n $(n \to \infty)$ are independent of g within each subclass

(ii) the distinction between subclasses only depends on $g(x)$ $(x \to \infty)$ and $g(x)$ $(x \downarrow 0)$. (0.6)

The subclasses on which (0.5–0.6) hold we call *universality classes* for the transformation F. For the probabilistic interpretation, this property means that the space-time scaling limit of the corresponding infinite system of interacting diffusions on $[0, \infty)$ has universal behavior independent of model parameters (see [DG3]; see also Part I Sections 0.3 and 0.6 and Part II Section 0.6).

One easily checks from (0.2–0.3) by explicit calculation (see Lemma 7 in Section 2.1) that for any $a \in (0, \infty)$

$$Fg_a = g_a \tag{0.7}$$

where

$$g_a(x) = ax \qquad (x \in [0, \infty)). \tag{0.8}$$

Thus all linear functions are *fixed points* of F. Because of this fact, the prime candidate for the limit function g^* in (0.5) is g_1.

Our program for this paper will be to answer the following questions:

(Q1). *Does F have fixed points in $\mathscr{H}$ other than $(g_a)_{a \in (0, \infty)}$?*

(Q2). *Are there $g \in \mathscr{H}$ for which $Fg = \lambda g$ with $\lambda \neq 1$ (i.e., does F have fixed shapes in $\mathscr{H}$)?*

(Q3). *Can one identify large subclasses of* $\mathcal{H}$ *on which* (0.5–0.6) *hold with pointwise convergence*? *For which of these subclasses is* $g^* = g_1$? *What is* d_n $(n \to \infty)$ *as a function of* $g(x)$ $(x \to \infty)$ *and* $g(x)$ $(x \downarrow 0)$?

(Q4). *Can the pointwise convergence be extended to convergence in a stronger norm*?

Questions (Q1)–(Q4) will be addressed in Sections 2–4. Our main results are formulated in Section 0.3.

Remark. Let us briefly recall that in Part I we had a similar transformation, but defined on $[0, 1]$. Namely, instead of (0.1) we had the class of functions $g: [0, 1] \to [0, \infty)$ satisfying $g(0) = g(1) = 0$, $g(x) > 0$ for $x \in (0, 1)$ and g Lipschitz continuous on $[0, 1]$, while in (0.2–0.3) both x and θ ran over $[0, 1]$. It turned out that the transformation on $[0, 1]$ had *no fixed point*. This already is a first indication that the behavior on $[0, 1]$ is qualitatively different from that on $[0, \infty)$. On the other hand, there was a unique g^* and a unique normalizing sequence $(d_n)_{n \geqslant 0}$ such that (0.5) was satisfied pointwise, namely $g^*(x) = x(1-x)$ and $d_n \sim n$ (see Part I Theorem 1). In other words, there was a *globally attracting* limit after appropriate scaling and this limit was in fact a *fixed shape* (i.e., $F^n g^* = d_n g^*$ for all n). We shall see that for the transformation on $[0, \infty)$ this type of global universality does not occur and that there are various different universality classes. Thus the situation is quite different.

0.3. *Main Theorems*

Theorems 1–6 below describe the universality classes of F.

(Q1). *Fixed points.*

THEOREM 1. *There are no fixed points in* $\mathcal{H}$ *other than* $(g_a)_{a \in (0, \infty)}$.

The proof of this theorem is quite delicate. In particular, we shall need the fact that F is convexity preserving, i.e., if g is convex then Fg is convex (see Proposition 3(f) in section 1.4 and Appendices A–B). This will allow us to show that every fixed point must be convex. Via a comparison with straight lines we shall then be able to show that fixed points cannot be strictly convex.

(Q2). *Fixed shapes.*

A fixed point gives rise to a trivial orbit. This brings up the following question. Can we find functions $g \in \mathcal{H}$ solving $Fg = \lambda g$ with $\lambda \neq 1$? More in particular, can we find a g for which there exists $d \to \lambda(d)$ such that $F(dg) = \lambda(d)\, g$ for all $d > 0$? This question is important, because for such a g the orbit would run through the multiples of one fixed shape.

Namely, $F^n g = \lambda_n g$ with $\lambda_n = \lambda^n(1)$, so that $g^* = g$ and $d_n = 1/\lambda_n$ in (0.5). In Part I we had this situation. However, it does not arise now:

THEOREM 2. *There are no $g \in \mathscr{H}$ solving $Fg = \lambda g$ with $\lambda \neq 1$.*

The proof of this theorem will come out of a comparison with parabolas.

One easily checks from (0.2–0.3) by explicit computation that if $g(x) = ax^2 + bx$ $(0 < a < 1,\ b \geqslant 0)$ then $Fg = (a/(1-a))\, g$ (see Lemma 5(c) in Section 1.2). Thus parabolas are fixed shapes. However, they fail to satisfy (0.1)(iv) and therefore are not in our class $\mathscr{H}$. Moreover, since $F^n g = (a/(1-na))\, g$ there is a finite n for which their orbit explodes. Still, parabolas will be very useful later on, when we shall use them as comparison objects.[1]

(Q3). *Convergence properties.*

I. We start with the simplest possible case, namely, one where (0.5–0.6) hold with a *constant* normalizing sequence. Denote by $C_c([0, \infty))$ the space of continuous functions on $[0, \infty)$ with the topology induced by the uniform convergence on compact subsets.

THEOREM 3. *If* $\lim_{x \to \infty} x^{-1} g(x) = a \in (0, \infty)$, *then*

$$\lim_{n \to \infty} F^n g = g_a \qquad \text{in } C_c([0, \infty)). \tag{0.9}$$

This result says that if $g(x)$ grows asymptotically like ax, then (0.5–0.6) hold pointwise and uniformly on compact subsets of $[0, \infty)$, with $g^* = g_1$ and $d_n \equiv a^{-1}$. Note here that it is $g(x)$ $(x \to \infty)$ which determines $(F^n g)(\theta)$ $(n \to \infty)$ for θ fixed. The limit in (0.9) depends on g through its asymptotic slope a but is otherwise universal.

II. Next we consider examples where (0.5–0.6) hold with *non-constant* normalizing sequences.

THEOREM 4. *Suppose that $g(x) \sim x^\alpha L(x)$ $(x \to \infty)$ with $\alpha \in (0, 2) \backslash \{1\}$ and L slowly varying at infinity. Let $(e_n)_{n \geqslant 1}$ be defined by*

$$\frac{1}{n} = \frac{g(e_n)}{e_n^2} \qquad (n \geqslant 1). \tag{0.10}$$

Then there exist constants $0 < K_1(\alpha) \leqslant K_2(\alpha) < \infty$ such that

$$g_{K_1} \leqslant \liminf_{n \to \infty} \frac{n}{e_n} F^n g \leqslant \limsup_{n \to \infty} \frac{n}{e_n} F^n g \leqslant g_{K_2} \ \textit{pointwise.} \tag{0.11}$$

[1] In the probabilistic context (explained in Section 0.6) $g(x) = ax^2 + bx$ makes perfectly good sense, but it belongs to a class of models that display completely different behavior (see [G]).

Remark. The solution $(e_n)_{n\geqslant 1}$ of (0.10) is in general not unique. However, all solutions are asymptotically equivalent (see Lemma 14 in Section 4.1). Therefore (0.11) identifies the asymptotic behavior of $F^n g$ uniquely.

The proof of Theorem 4 is based on an approximation by parabolas. Unfortunately, we are not able to narrow down the constants. Our best estimates for K_1 and K_2 are

$$\begin{aligned} &\alpha\in(0,1)\colon K_1\geqslant 1,\ K_2\leqslant(2-\alpha)^{1/(2-\alpha)}\\ &\alpha\in(1,2)\colon K_1\geqslant(2-\alpha)^{1/(2-\alpha)},\ K_2\leqslant 1. \end{aligned}\tag{0.12}$$

We conjecture that

$$\lim_{n\to\infty}\frac{n}{e_n}F^n g=g_K \text{ pointwise, with } K=(\alpha!)^{1/(2-\alpha)}\,2^{(1-\alpha)/(2-\alpha)}.\tag{0.13}$$

However, as we shall see in Section 4.1, the proof of this result would depend on solving some delicate problems related to scaling properties of certain kernel iterates.

If $L\equiv 1$, then $e_n\sim n^{1/(2-\alpha)}$ and so $F^n g\asymp n^{-(1-\alpha)/(2-\alpha)}g_1$. Thus, $n\to F^n g$ moves down when $\alpha\in(0,1)$ and up when $\alpha\in(1,2)$. The rate at which this happens depends on g through its asymptotic growth exponent α but is otherwise universal. Note that *locally* $F^n g$ gets close to linear (assuming that the conjecture in (0.13) is true). Still, it keeps on moving with n because *globally* it is not linear (recall that straight lines are fixed points). In other words, the attracting orbit is a rotating line that attracts on an expanding domain but not as a whole. This is an interesting phenomenon coming from the non-linearity of the map F. What Theorem 4 says in essence is that $F^n g$ is determined by what g looks like in the neighborhood of $e_n=e_n(g)$ and that it "moves by curvature" (see Proposition 3(b) in Section 1.4).

(Q4). *Finer topologies.*

I. Equation (0.9) gives us a good description of the orbit away from 0 and ∞. In order to capture what may happen close to these boundaries, we next introduce two finer topologies τ_0 and τ_∞ on subspaces of $C_c([0,\infty))$. Namely, define the seminorms

$$\|f\|_A=\sup_{x\in A}\left|\frac{f(x)}{g_1(x)}\right| \qquad (A\subset(0,\infty),\ f\in C_c([0,\infty)),\ \|f\|_A<\infty)\tag{0.14}$$

and let

$$\begin{aligned} &\text{(i)}\quad \tau_0 \text{ is generated by } \{\|\cdot\|_{(0,N)},\ N<\infty\}\\ &\text{(ii)}\quad \tau_\infty \text{ is generated by } \{\|\cdot\|_{(\delta,\infty)},\ \delta>0\}. \end{aligned}\tag{0.15}$$

In (0.14) we have chosen g_1 as the "natural" reference function, because in the situation of Theorem 3 we have $g^* = g_1$ in (0.5). Convergence in τ_0 means uniform convergence of $(F^n g)/g_1$ in a neighborhood of 0. Convergence in τ_∞ means uniform convergence of $(F^n g)/g_1$ in a neighborhood of ∞.

THEOREM 5. *Suppose that* $\lim_{x \to \infty} x^{-1} g(x) = a \in (0, \infty)$. *Then the convergence in* (0.9) *holds*:

(i) *in* τ_∞ *always*
(ii) *in* τ_0 *if* $\liminf_{x \downarrow 0} x^{-2} g(x) > 0$
(iii) *not in* τ_0 *if* $\limsup_{x \downarrow 0} x^{-2} g(x) = 0$.

We see from Theorem 5 that the two boundaries behave differently. In particular, to get convergence in the τ_0-topology it is apparently critical that g stays above a *quadratic* in a neighborhood of 0. We shall see in Section 3 that, if and only if g has this property, $F^n g$ develops a *strictly positive* slope at 0 after finitely many iterations and this slope tends to a as $n \to \infty$, which is the slope of the limit g_a. The same phenomenon was found in Part I Theorem 2. Thus, F has a remarkable behavior near the boundary at 0.

II. The analogue of Theorem 5 corresponding to Theorem 4 reads as follows.

THEOREM 6. *Assume g as in Theorem* 4.

(i) *The convergence in* (0.11) *strengthens to*

$$\lim_{\delta, \varepsilon \downarrow 0} \limsup_{n \to \infty} \left\| \frac{n}{e_n} F^n g - g_K \right\|_{(\delta, \varepsilon e_n)} \leqslant \| g_{K_1} - g_{K_2} \|_{(0, \infty)}. \tag{0.16}$$

(ii) *If* $\liminf_{x \downarrow 0} x^{-2} g(x) > 0$, *then* (0.16) *holds with* $\delta = 0$.
(iii) *If* $\limsup_{x \downarrow 0} x^{-2} g(x) = 0$, *then* (0.16) *does not hold with* $\delta = 0$.

Thus, the behavior in the τ_0-topology carries over, while the convergence at the other end is *uniform on a growing interval of length* $o(e_n)$. If the conjecture (0.13) is true, then $K_1 = K_2 = K$ and the limit in (0.16) is 0. Incidentally, the latter cannot be extended to convergence in the τ_∞-topology, since it can be shown that F does not change the behavior at infinity, i.e., $\lim_{\theta \to \infty} (Fg)(\theta)/g(\theta) = 1$ for all g as in Theorem 4 (see Lemma 21 in Section 4.6). Consequently, $\|(n/e_n) F^n g - g_K\|_{(\delta, \infty)} \geqslant 1$ for all $n \geqslant 1$ and $\delta > 0$.

Theorems 1–6 are proved Sections 2–4.

0.4. *A Generalization*

For the application of (0.1–0.3) to the infinite system of interacting diffusions (as explained in [DG3] and Part I Sections 0.3 and 0.6) it is actually important to slightly generalize the transformation F. For $c>0$, let $\nu_\theta^{g,c}$ and F_c be defined by

$$\nu_\theta^{g,c} = \nu_\theta^{(1/c)\,g}$$
$$(F_c g)(\theta) = \left(cF\left(\frac{1}{c}g\right)\right)(\theta) = \int_0^\infty g(x)\,\nu_\theta^{g,c}(dx). \tag{0.17}$$

Pick any sequence $(c_k)_{k\geqslant 0}\subset(0,\infty)$ and define the *inhomogeneous* composition

$$F^{(n)} = F_{c_{n-1}}\circ\cdots\circ F_{c_0} \qquad (n\geqslant 0) \tag{0.18}$$

$(F^{(0)}=Id)$. Then we can ask the same questions (Q1)–(Q4) for the orbit

$$\{F^{(n)}g\}_{n=0}^\infty \tag{0.19}$$

replacing (0.4). The answers will now also depend on c_k $(k\to\infty)$.

The parameter c_k plays the role of the interaction strength between diffusing components that are at hierarchical distance k (see Part I Section 0.3). It turns out that the behavior of the system is similar as in the case $c_k\equiv 1$, as long as $1/c_k$ is not summable.

THEOREM 7. *Assume g as in Theorem* 4. *Define*

$$\sigma_n = \sum_{k=0}^{n-1}\frac{1}{c_k} \tag{0.20}$$

and assume that $\sigma_\infty=\infty$. *Let* $(e_n)_{n\geqslant 1}$ *be defined by*

$$\frac{1}{\sigma_n} = \frac{g(e_n)}{e_n^2} \qquad (n\geqslant 1). \tag{0.21}$$

Then for the same K_1, K_2 *as in* (0.11)

$$g_{K_1} \leqslant \liminf_{n\to\infty}\frac{\sigma_n}{e_n}F^{(n)}g \leqslant \limsup_{n\to\infty}\frac{\sigma_n}{e_n}F^{(n)}g \leqslant g_{K_2} \text{ pointwise.} \tag{0.22}$$

The extension to the finer topologies introduced in (0.14–0.15) *is similar as in Theorem* 6: *simply replace* (F^n, n) *by* $(F^{(n)}, \sigma_n)$.

This result shows that apparently the roles of g and $(c_k)_{k \geqslant 0}$ can be separated. The same situation was found in Part I.

The meaning of the restriction $\sigma_\infty = \infty$ can be found in Part I. Roughly, if $\sigma_\infty < \infty$ then $F^{(n)}g$ converges to a limit that depends on g and therefore does not meet the universality requirement in (0.6)(i).

Theorem 7 is proved in Section 5.

0.5. *A Larger Domain*

The class where the transformation F is defined can actually be chosen much larger than $\mathscr{H}$ given by (0.1). Namely, let $\mathscr{H}' \supset \mathscr{H}$ be the class of functions g: $[0, \infty) \to [0, \infty)$ satisfying

$$\begin{aligned} &\text{(i)} && g \text{ measurable} \\ &\text{(ii)} && g^{-1} \text{ locally integrable on } (0, \infty) \\ &\text{(iii)} && g^{-1} \text{ not integrable at } 0 \\ &\text{(iv)} && \lim_{x \to \infty} x^{-2} g(x) = 0. \end{aligned} \tag{0.23}$$

Then the analogue of Lemma 1 reads:

LEMMA 2. (i) *F is well defined on $\mathscr{H}'$.*

(ii) *$F\mathscr{H}' \subset \mathscr{H}$.*

(iii) *For all $g \in \mathscr{H}'$ the function $\theta \to (Fg)(\theta)$ is C^∞ on $(0, \infty)$.*

The class $\mathscr{H}$ is the "natural" class for the probabilistic interpretation of (0.1–0.3) (see [Sh]). Still, F as a transformation makes sense on $\mathscr{H}'$. However, Lemma 2(ii) says that after one iteration one falls back onto the original class $\mathscr{H}$. Therefore the extension to $\mathscr{H}'$ does not survive the transformation.

Lemma 2(iii) shows that F has a strong smoothing property. Thus, after one iteration one in fact falls back onto a class much smaller than $\mathscr{H}$.

0.6. *Probabilistic Background*

We close this Section 0 with a brief explanation of how (0.1–0.3) arise naturally from a space-time scaling analysis of a system of interacting diffusions on $[0, \infty)$. The remarks are intended to motivate the reader and are independent of the rest of the paper.

Consider the following system of N coupled stochastic differential equations:

$$\begin{aligned} dX_i^N(t) &= \frac{1}{N} \sum_{j=1}^{N} [X_j^N(t) - X_i^N(t)]\, dt + \sqrt{2g(X_i^N(t))}\, dW_i(t) \\ X_i^N(0) &= \theta \qquad (i = 1, ..., N). \end{aligned} \tag{0.24}$$

Here the $W_i(t)$ are independent standard Brownian motions, and g is any function satisfying (0.1). Eq. (0.24) says that the $X_i^N(t)$ are diffusions on $[0, \infty)$, with a local diffusion rate given by g and with a *mean field* interaction that makes each diffusion drift towards the current empirical mean of the whole system. The conditions on g in (0.1) are natural: $g(0)=0$, the Lipschitz property of g at 0, and $g(x)>0$ for $x>0$ guarantee that the diffusions live on $[0, \infty)$; the local Lipschitz property of g on $(0, \infty)$ guarantees that (0.24) has a unique strong solution.

Now, it can be shown that as $N \to \infty$ (see [DG4])

$$\mathscr{L}\{(X_1^N(t), ..., X_k^N(t))_{t \geqslant 0}\} \Rightarrow [\mathscr{L}\{(Z_\theta^g(t))_{t \geqslant 0}\}]^{\otimes k} \qquad (k \geqslant 1) \quad (0.25)$$

($\mathscr{L}$ means law and $\Rightarrow$ means weak convergence), where the limit process is the k-fold product of the unique strong solution of the 1-component stochastic differential equation

$$\begin{aligned} dZ(t) &= [\theta - Z(t)]\, dt + \sqrt{2g(Z(t))}\, dW(t) \\ Z(0) &= \theta. \end{aligned} \qquad (0.26)$$

In other words, each component decouples from the rest ("propagation of chaos") but retains a drift towards θ, the *initial* empirical mean of the whole system.

How can we understand (0.25–0.26)? The heuristics is as follows. For large N the empirical mean fluctuates on a slow time scale. It will stay constant on the original time scale in the limit as $N \to \infty$, i.e., fixed at its initial value θ. This explains why each component of (0.24) has as limit dynamics (0.26). The behavior of $Z_\theta^g(t)$ as $t \to \infty$ in (0.25) will be given by the *equilibrium solution* of (0.26). The link with the probability measure ν_θ^g, defined in (0.2) and appearing in the definition of F in (0.3), is that ν_θ^g is precisely the marginal law of this equilibrium.

Next consider the quantity

$$\hat{X}^N(t) = \frac{1}{N} \sum_{i=1}^{N} X_i^N(tN), \qquad (0.27)$$

i.e., the empirical mean but speeded up proportionally to the size of the system. It follows from (0.24) that

$$d\hat{X}^N(t) = \frac{1}{N} \sum_{i=1}^{N} \sqrt{2g(X_i^N(tN))}\, dW_i(tN). \qquad (0.28)$$

With the help of a martingale argument it can be shown that as $N \to \infty$ (see [DG4])

$$(\hat{X}^N(t))_{t \geq 0} \Rightarrow (Z^{Fg}(t))_{t \geq 0}, \tag{0.29}$$

where the limit process is the unique strong solution of

$$\begin{aligned} dZ(t) &= \sqrt{2(Fg)(Z(t))}\, dW(t) \\ Z(0) &= \theta, \end{aligned} \tag{0.30}$$

i.e., a diffusion without drift but with a *new* diffusion function Fg.

How can we understand (0.29–0.30)? This result becomes quite natural when we remember the definition of F in (0.3), which says that $(Fg)(\theta)$ is the average of $g(x)$ under $\nu_\theta^g(dx)$. Namely, in these terms (0.29–0.30) simply say that the empirical mean $\hat{X}^N(t)$ $(N \to \infty)$ has a local diffusion rate that is the average of the local diffusion rate for the single components under their equilibrium law. Indeed, this is plausible from (0.28) when we observe that $\mathscr{L}\{N^{-1/2} W_i(tN)\} = \mathscr{L}\{W_i(t)\}$ and $X_i^N(tN) \Rightarrow \nu_\theta^g$ $(N \to \infty)$.

Thus we have explained why the transformation F is *naturally induced* by the space-time scaling in (0.27) in the limit as $N \to \infty$. The higher iterates of F come into play when, instead of (0.24), one considers an infinite system with an infinite hierarchy of components and interactions. This gives rise to an *infinite hierarchy of space-time scales*, which are described by $F^n g$ $(n \geq 1)$ (see [DG3-4]). More precisely, $F^n g$ appears as the diffusion function associated with "block averages on space-time scale n," i.e., block averages of N^n components observed at time tN^n (similarly as in (0.27)). These are described by an equation like (0.30) with Fg replaced by $F^n g$. The universality properties of F formulated in Theorems 1–6 in Section 0.3 mean that the fluctuations of these block averages show universal behavior as $n \to \infty$.

1. PROOF OF LEMMAS 1, 2, AND SOME KEY PROPERTIES OF F

In this section we give the proof of Lemmas 1 and 2 and collect some major tools (Propositions 1–4 below) that will be needed for the proof of Theorems 1–6 in Sections 2–4.

1.1. *Reformulation of F*

It will be expedient to rewrite the definition of F into a form more suitable for manipulations. Namely, define

$$\mu_\theta^g(x) = \frac{1}{g(x)} \exp\left[-\int_\theta^x \frac{y-\theta}{g(y)}\, dy \right] \qquad (x, \theta > 0) \tag{1.1}$$

and write (0.3) as

$$(Fg)(\theta) = \frac{\int_0^\infty g(x)\, \mu_\theta^g(x)\, dx}{\int_0^\infty \mu_\theta^g(x)\, dx} \qquad (\theta > 0). \tag{1.2}$$

The integrand in the numerator has a nice shape property:

LEMMA 3. *For all* $\theta \in (0, \infty)$

$$\begin{gathered} \frac{\partial}{\partial x}\left[g(x)\, \mu_\theta^g(x) \right] = (\theta - x)\, \mu_\theta^g(x) \\ g(\theta)\, \mu_\theta^g(\theta) = 1. \end{gathered} \tag{1.3}$$

Hence $x \to g(x)\, \mu_\theta^g(x)$ *is increasing on* $(0, \theta)$, *decreasing on* (θ, ∞) *and has a maximum value* 1 *at the point* θ. *Moreover*, $\lim_{x \downarrow 0} g(x)\, \mu_\theta^g(x) = 0$.

Proof. Immediate from (1.1). The first relation in (1.3) expresses the fact that μ_θ^g is the density of the equilibrium of (0.26) up to normalization. The last statement is a direct consequence of (0.1)(i, iii), which imply that g^{-1} is not integrable at 0. ▮

To prove Lemmas 1–2 we shall need the following monotonicity property:

LEMMA 4. *For any* $0 \leqslant a \leqslant \theta \leqslant b \leqslant \infty$, *if* $g_1 \leqslant g_2$ *on* $[a, b]$ *then*

$$\begin{aligned} &\text{(a)} \quad g_1 \mu_\theta^{g_1} \leqslant g_2 \mu_\theta^{g_2} \text{ on } [a, b] \\ &\text{(b)} \quad \int_a^b \mu_\theta^{g_1}(x)\, dx \geqslant \int_a^b \mu_\theta^{g_2}(x)\, dx. \end{aligned} \tag{1.4}$$

Proof. Part (a) is evident from (1.1), because $-g_1^{-1} \leqslant -g_2^{-1}$ on $[a, b]$. Part (b) follows from Part (a) and the representations

$$\begin{aligned} \int_\theta^b \mu_\theta^g(x)\, dx &= \frac{1 - g(b)\, \mu_\theta^g(b)}{b - \theta} + \int_\theta^b \frac{1 - g(x)\, \mu_\theta^g(x)}{(x - \theta)^2}\, dx \qquad (b > \theta) \\ \int_a^\theta \mu_\theta^g(x)\, dx &= \frac{1 - g(a)\, \mu_\theta^g(a)}{\theta - a} + \int_a^\theta \frac{1 - g(x)\, \mu_\theta^g(x)}{(\theta - x)^2}\, dx \qquad (a < \theta), \end{aligned} \tag{1.5}$$

which are obtained by partial integration using (1.3). (First exclude an ε-neighborhood of θ to avoid the pole of $x \to (x - \theta)^{-1}$ and then let $\varepsilon \downarrow 0$.) ▮

Note that the inequalities in (1.4)(a–b) go in the *opposite* direction. This will be important later on.

1.2. *Proof of Lemmas* 1 *and* 2

Lemma 2 is a generalization of Lemma 1 (compare (0.1) and (0.23)), so we need only prove the former.

We begin by explaining conditions (0.23)(i–iv). It is clear that (0.23)(i–ii) are the minimal conditions required for the integrals in (0.2–0.3) to make sense. To explain (0.23)(iv) we prove the following.

LEMMA 5. *Assume* (0.23)(i–ii).

(a) *If* $g(x) \geqslant x^2$ *then* Fg *is* ∞ *on* $(0, \infty)$.

(b) *If* $g(x) \leqslant ax^2 + b$ $(0 \leqslant a < 1, 0 < b < \infty)$ *then* Fg *is finite on* $[0, \infty)$.

(c) *If* $g(x) = ax^2$ $(0 < a < 1)$ *then* $(Fg)(x) = (a/(1-a))\, x^2$.

(d) *If* $g(x) = o(x^2)$ $(x \to \infty)$ *then* $(Fg)(x) = o(x^2)$ $(x \to \infty)$.

Proof. Let $N(\theta)$ and $D(\theta)$ denote the numerator resp. denominator of (1.2).

(a) Insert $g(x) \geqslant x^2$ into $N(\theta)$ to find, using Lemma 4(a),

$$\begin{aligned} N(\theta) &\geqslant \int_0^\infty \exp\left[-\int_\theta^x \frac{y-\theta}{y^2}\, dy\right] dx \\ &= \int_0^\infty \frac{\theta}{x} \exp\left(1 - \frac{\theta}{x}\right) dx \\ &\geqslant \int_\theta^\infty \frac{\theta}{x}\, dx \\ &= \infty. \end{aligned} \tag{1.6}$$

Similarly, using Lemma 4(b),

$$\begin{aligned} D(\theta) &\leqslant \int_0^\infty \frac{1}{x^2} \exp\left[-\int_\theta^x \frac{y-\theta}{y^2}\, dy\right] dx \\ &= \int_0^\infty \frac{\theta}{x^3} \exp\left(1 - \frac{\theta}{x}\right) dx \\ &= \frac{e}{\theta} \int_0^\infty z e^{-z}\, dz \\ &< \infty. \end{aligned} \tag{1.7}$$

(d) Assume $g(x) = o(x^2)$ $(x \to \infty)$. Then for every $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that $g(x) \leqslant \varepsilon x^2$ for all $x \geqslant N$. Hence, using Lemma 4(a), we can estimate for $\theta \geqslant N$

$$\begin{aligned} N(\theta) &\leqslant N + \int_N^\infty \exp\left[-\int_\theta^x \frac{y-\theta}{\varepsilon y^2}\,dy\right] dx \\ &= N + e^{1/\varepsilon} \int_N^\infty \left(\frac{\theta}{x}\right)^{1/\varepsilon} \exp\left(-\frac{\theta}{\varepsilon x}\right) dx \\ &= N + \theta e^{1/\varepsilon} \varepsilon^{(1/\varepsilon)-1} \int_0^{\theta/\varepsilon N} z^{(1/\varepsilon)-2} e^{-z}\,dz \end{aligned} \tag{1.8}$$

and similarly, using Lemma 4(b),

$$\begin{aligned} D(\theta) &\geqslant \int_N^\infty \frac{1}{\varepsilon x^2} \exp\left[-\int_\theta^x \frac{y-\theta}{\varepsilon y^2}\,dy\right] dx \\ &= e^{1/\varepsilon} \int_N^\infty \frac{1}{\varepsilon x^2} \left(\frac{\theta}{x}\right)^{1/\varepsilon} \exp\left(-\frac{\theta}{\varepsilon x}\right) dx \\ &= \theta^{-1} e^{1/\varepsilon} \varepsilon^{1/\varepsilon} \int_0^{\theta/\varepsilon N} z^{1/\varepsilon} e^{-z}\,dz. \end{aligned} \tag{1.9}$$

By combining (1.8–1.9) and letting $\theta \to \infty$ for fixed ε, we get

$$\limsup_{\theta \to \infty} \theta^{-2} \frac{N(\theta)}{D(\theta)} \leqslant \frac{e^{1/\varepsilon} \varepsilon^{(1/\varepsilon)-1} \Gamma((1/\varepsilon)-1)}{e^{1/\varepsilon} \varepsilon^{1/\varepsilon} \Gamma((1/\varepsilon)+1)} = \frac{\varepsilon}{1-\varepsilon}. \tag{1.10}$$

Now let $\varepsilon \downarrow 0$ to get the claim.

(b) Since $\nu_0^g = \delta_0$ (recall (0.2)), we have $(Fg)(0) = 0$. Next, fix $\theta > 0$. Since g^{-1} is locally integrable on $(0, \infty)$, the part of the integrals in $N(\theta)$ and $D(\theta)$ where the integrating variable x is restricted to a compact subset of $(0, \infty)$ is finite. On the other hand, the estimates in (1.8–1.9) show that no divergence can occur at $x = \infty$ as soon as $g(x) \leqslant ax^2$ for large x with $(1/a) - 2 > -1$, i.e., $a < 1$ (set $N = 0$ and $\varepsilon = a$ in (1.8–1.9)). Morover, in $N(\theta)$ no divergence can occur at $x = 0$ because $g\mu_\theta^g \leqslant 1$ by Lemma 3.

(c) The same type of computation as in (1.8–1.9) yields $N(\theta) = \theta e^{1/a} a^{(1/a)-1} \Gamma((1/a)-1)$ and $D(\theta) = \theta^{-1} e^{1/a} a^{1/a} \Gamma((1/a)+1)$ (again set $N = 0$ and $\varepsilon = a$ in (1.8–1.9)). This gives the claim because $\Gamma((1/a)+1)/\Gamma((1/a)-1) = (1-a)/a^2$. ∎

We see from Lemmas 5(a, c) that without (0.23)(iv) the orbit $\{F^n g\}_{n=0}^\infty$ would be ill defined. Namely, the nth iterate of the map $a \to a/(1-a)$ is $a \to a/(1-na)$, and this explodes when $n > 1/a$. Combining Lemmas 4 and 5(a, c), one can show that (0.23)(iv) is essentially a necessary condition to avoid explosion. Lemmas 5(b, d) show that it is a sufficient condition.

To understand (0.23)(iii), we look a bit closer at the behavior of F close to the left boundary.

LEMMA 6. *Assume* (0.23)(i–ii, iv).

(a) $\lim_{\theta \downarrow 0} (Fg)(\theta) = c < \infty$ *exists with* $c = 0$ *iff* $x \to 1/g(x)$ *is not integrable at* 0.

(b) *If* $x \to 1/g(x)$ *is not integrable at* 0, *then* $\lim_{\theta \downarrow 0} \theta^{-1}(Fg)(\theta) = c' < \infty$ *exists with* $c' = 0$ *iff* $x \to x/g(x)$ *is not integrable at* 0.

Proof. Same as Propositions 4 and 5 in Part I. The proof relies on Lemma 3 in combination with a sequence of explicit estimates that are given in Part I Section 2.2. ∎

We can now give the proof of Lemma 2. Lemma 2(iii) is the same as Theorem 5 in Part I and can be proved by the same method as in Part I Section 2.6. This method used an explicit representation of the formal derivatives of Fg. The same representation holds here. We omit the details. Lemmas 5(d) and 6 imply Lemmas 2(i–ii). Indeed, if g satisfies (0.23)(i–iv) then:

(1) F is well defined on $\mathscr{H}'$, because of (0.23)(i–ii, iv).

(2) $(Fg)(0) = 0$ because $\nu_0^g = \delta_0$ (recall (0.2)), so Fg satisfies (0.1)(i).

(3) (0.23)(iii) and Lemma 6 give that Fg is continuous at 0 and has a finite slope at 0, so in particular Fg is Lipschitz at 0. Since Lemma 2(iii) implies that Fg is locally Lipschitz on $(0, \infty)$, it follows that Fg satisfies (0.1)(iii).

(4) (0.23)(iv) and Lemma 5(d) show that Fg satisfies (0.1)(iv).

(5) (0.23)(ii) obviously implies that Fg satisfies (0.1)(ii).

1.3. *Moment Relations*

The following four relations play a crucial role in the paper:

PROPOSITION 1. *For all* $g \in \mathscr{H}$ *and* $\theta \in [0, \infty)$

$$
\begin{aligned}
&\text{(a)} \quad \int_0^\infty \nu_\theta^g(dx) = 1\\
&\text{(b)} \quad \int_0^\infty x\nu_\theta^g(dx) = \theta\\
&\text{(c)} \quad \int_0^\infty x^2\nu_\theta^g(dx) = \theta^2 + (Fg)(\theta)\\
&\text{(d)} \quad \int_0^\infty g(x)\, \nu_\theta^g(dx) = (Fg)(\theta).
\end{aligned}
\tag{1.11}
$$

Proof. (a, d) are (0.2–0.3). It is straightforward to check (b–c) by explicit calculation. Another way is via Itô's formula using that $\nu_\theta^g(dx)$ is the equilibrium of (0.26). The derivation along this line also makes it clear that what matters for (b–c) is not the explicit form of $\nu_\theta^g(dx)$ but rather its equilibrium property. Note that (b) shows that linear functions are fixed points of F. ∎

For $g \in \mathcal{H}$, define the probability kernel on $[0, \infty) \times [0, \infty)$

$$K_g(x, dy) = \nu_x^g(dy) \tag{1.12}$$

and the *inhomogeneous* composition

$$K_g^{(n)} = K_{F^{n-1}g} \circ \cdots \circ K_{F^0 g} \qquad (n \geqslant 1). \tag{1.13}$$

($F^0 = Id$, $K_g^{(1)} = K_g$, $Fg = K_g\, g$.) In terms of these quantities Proposition 1 can be iterated:

PROPOSITION 2. *For all $g \in \mathcal{H}$, $\theta \in [0, \infty)$ and $n \geqslant 1$*

$$\begin{aligned}
&\text{(a)} \quad \int_0^\infty K_g^{(n)}(\theta, dy) = 1 \\
&\text{(b)} \quad \int_0^\infty y K_g^{(n)}(\theta, dy) = \theta \\
&\text{(c)} \quad \int_0^\infty y^2 K_g^{(n)}(\theta, dy) = \theta^2 + n(F^n g)(\theta) \\
&\text{(d)} \quad \int_0^\infty g(y)\, K_g^{(n)}(\theta, dy) = (F^n g)(\theta).
\end{aligned} \tag{1.14}$$

Proof. Combine (1.12–1.13) with Proposition 1: (a–b, d) are immediate from (1.11)(a–b, d); (c) is obtained by combining (1.11)(c–d). ∎

1.4. *Some Key Properties of F*

We formulate some key properties of the transformation F that will be needed later on.

PROPOSITION 3. *On $\mathcal{H}$ the following properties hold*:

(a) $g_1 \leqslant g_2 \Rightarrow Fg_1 \leqslant Fg_2$, *with strict inequality everywhere on* $(0, \infty)$ *unless* $g_1 \equiv g_2$.

(b) g *convex* (*concave*) $\Rightarrow Fg \geqslant g$ ($Fg \leqslant g$), *with strict inequality everywhere on* $(0, \infty)$ *unless g is linear.*

(c) *F is continuous in the metric induced by*

$$\|f\| = \sup_{x \in [0, \infty)} \frac{|f(x)|}{1+x^2}, \tag{1.15}$$

i.e., if $\lim_{n\to\infty} \|g_n - g\| = 0$ *then* $\lim_{n\to\infty} \|Fg_n - Fg\| = 0$.

(d) *F is a contraction in the global Lipschitz norm, i.e.,* $L(Fg) \leqslant L(g)$ *where L is defined by* $Lg = \sup_{x, y \in (0, \infty), x \neq y} |g(x) - g(y)|/|x-y|$.

(e) *g increasing (decreasing)* $\Rightarrow$ *Fg increasing (decreasing).*

(f) *g convex (concave)* $\Rightarrow$ *Fg convex (concave).*

Proof. (a) Immediate from Lemma 4 after setting $a=0$ and $b=\infty$. The second half of the statement is an easy consequence of the fact that F acts globally, meaning that $(Fg)(\theta)$ depends on the value of $g(x)$ for all $x>0$.

(b) Jensen's inequality applied to (0.3). Use the identity $\int_0^\infty x\nu_\theta^g(dx) = \theta$, which is Proposition 1(b).

(c) First we show that F is pointwise continuous under monotone convergence. Indeed, let $g_n \uparrow g$ in $\mathscr{H}$. We know from Lemma 4 that $g \to g\mu_\theta^g$ is monotone increasing. Hence, in (1.2) the numerator of $(Fg_n)(\theta)$ converges to the numerator of $(Fg)(\theta)$. On the other hand, from (1.5) and the last statement in Lemma 3 we know that

$$\int_0^\infty \mu_\theta^g(x)\,dx = \int_0^\infty \frac{1-g(x)\,\mu_\theta^g(x)}{(x-\theta)^2}\,dx + \frac{1}{\theta}. \tag{1.16}$$

Therefore, in (1.2) also the denominator of $(Fg_n)(\theta)$ converges to the denominator of $(Fg)(\theta)$. Thus, $Fg_n \to Fg$ pointwise. A similar argument works when $g_n \downarrow g$ in $\mathscr{H}$.

Next we show that F is pointwise continuous under convergence in $\|\cdot\|$. Indeed, assume that $\|g_n - g\| \to 0$ in $\mathscr{H}$. Then both

$$\begin{aligned} g_n^- &= \inf_{m \geqslant n} g_m \\ g_n^+ &= \sup_{m \geqslant n} g_m. \end{aligned} \tag{1.17}$$

are in $\mathscr{H}$ for all n (see (0.1)). Clearly, $g_n^- \uparrow g$ and $g_n^+ \downarrow g$. Hence, $Fg_n^- \uparrow Fg$ and $Fg_n^+ \downarrow Fg$. But $g_n^- \leqslant g_n \leqslant g_n^+$ and therefore $Fg_n^- \leqslant Fg_n \leqslant Fg_n^+$, so we get $Fg_n \to Fg$ pointwise.

Finally, since $\|Fg_n - Fg\| \leqslant \|Fg_n^+ - Fg_n^-\| \downarrow 0$ we get the claim.

(d) See Part I Section 2.7. See also [DG2] Lemma 2.2 and Equation (2.58).

(e–f) Implied by Propositions 4(e′–f′) below. ∎

The following proposition is a version of Proposition 3 for the kernel K_g defined in (1.12). This version will play an important role in the proof of Theorems 1–2, 4 and 6. The advantage of working with K_g is that $f \to K_g f$ is linear, while $g \to Fg = K_g g$ is not.

PROPOSITION 4. *For* $g \in \mathscr{H}$ *and* $f, f_1, f_2 \in \bigcap_{\theta \in [0,\infty)} L^1([0,\infty); \nu_\theta^g)$:

(a′) $f_1 \leqslant f_2 \Rightarrow K_g f_1 \leqslant K_g f_2$, *with strict inequality everywhere on* $(0, \infty)$ *unless* $f_1 \equiv f_2$.

(b′) f *convex* (*concave*) $\Rightarrow K_g f \geqslant f$ ($K_g f \leqslant f$), *with strict inequality everywhere on* $(0, \infty)$ *unless* f *is linear.*

(e′) f *increasing* (*decreasing*) $\Rightarrow K_g f$ *increasing* (*decreasing*).

(f′) f *convex* (*concave*) $\Rightarrow K_g f$ *convex* (*concave*).

For $g_1, g_2 \in \mathscr{H}$ *and* $f \in \bigcap_{\theta \in [0,\infty)} L^1([0,\infty); \nu_\theta^{g_2})$:

(g′) f *convex* (*concave*), $g_1 \leqslant g_2 \Rightarrow K_{g_1} f \leqslant K_{g_2} f$ ($K_{g_1} f \geqslant K_{g_2} f$).

Proof. (a′–b′) Trivial extensions of (a–b).

(e′–g′) Deferred to Appendices A–C. The proof is analytic but lengthy. ∎

Note that, by Propositions 1(a, c), for all $g \in \mathscr{H}$

$$\{f\colon [0,\infty) \to \mathbb{R} \text{ measurable: } \|f\| < \infty\} \subset \bigcap_{\theta \in [0,\infty)} L^1([0,\infty); \nu_\theta^g), \qquad (1.18)$$

with $\|\cdot\|$ defined in (1.15). Thus Proposition 4 applies to a very large class of functions.

2. PROOF OF THEOREMS 1–3

The proof will run via a comparison with straight lines and parabolas, which are fixed points resp. fixed shapes. Propositions 3–4 will turn out to be crucial.

2.1. *Proof of Theorem* 1

Each $g \in \mathscr{H}$ can be trivially extended to a function on $\mathbb{R}$ by defining $g(x) = 0$ for $x < 0$ (recall (0.1)). The same can be done for Fg, and we shall henceforth view $\mathscr{H}$ and F as trivially extended in this way.

We have seen in Section 0.2 that all the linear functions $(g_a)_{a \in (0,\infty)}$ are fixed points of F in $\mathscr{H}$. (This fact is immediate from Proposition 1(b).)

To prove Theorem 1 it will be useful to temporarily enlarge $\mathscr{H}$ by adding all functions that fall in $\mathscr{H}$ after a *shift*, i.e., the class

$$\mathscr{H}_s = \{g\colon g(\cdot - b) \in \mathscr{H} \text{ for some } b \in \mathbb{R}\}. \tag{2.1}$$

We can extend F from $\mathscr{H}$ to $\mathscr{H}_s$ in the obvious way. Namely, for $g \in \mathscr{H}$ and $b \in \mathbb{R}$ define

$$\begin{aligned} \mu_\theta^{g(\cdot - b)}(x) &= \mu_{\theta - b}^{g(\cdot)}(x - b) \qquad &(x, \theta > b) \\ &= 0 \qquad &(x \leqslant b \text{ or } \theta \leqslant b) \end{aligned} \tag{2.2}$$

and use this extended definition in (1.1–1.2). In other words, F commutes with the shift, written $F(g(\cdot - b)) = (Fg)(\cdot - b)$.

The reason for looking at the larger class $\mathscr{H}_s$ is that now also the shifted linear functions become fixed points (in $\mathscr{H}_s$), and this will turn out to be useful for the proof of Theorem 1 (see below).

LEMMA 7. *All* $(g_{a,b})_{a \in (0,\infty),\, b \in \mathbb{R}}$ *defined by*

$$g_{a,b}(x) = a(x - b)\, 1_{\{x \geqslant b\}} \qquad (x \in \mathbb{R}) \tag{2.3}$$

are fixed points of F *in* $\mathscr{H}_s$.

Proof. Immediate from Propositions 1(a–b) or (2.1–2.2). ∎

Furthermore, Lemmas 3–6 and Propositions 1–4 carry over in the obvious way.

The proof of Theorem 1 now comes in two steps.

Step 1. All fixed points are convex.

Proof. Suppose that $Fg = g$. Let $f(x) = x^2$. Then, by Propositions 1(c–d),

$$([K_g]^n f)(\theta) = \theta^2 + ng(\theta) \qquad (n \geqslant 0). \tag{2.4}$$

Since f is convex, it follows from Proposition 4(f′) that the r.h.s. of (2.4) is convex for all $n \geqslant 0$. Divide by n and let $n \to \infty$ to get the claim. ∎

Step 2. Convex fixed points must be linear.

Proof. The proof is by contradiction, through the following geometrical argument. Suppose that $Fg = g$. Then g is convex by Step 1. Suppose that g is not linear. Then somewhere on $(0, \infty)$ it is strictly convex and so there exist $a, b > 0$ and $x_0 > 0$ such that

$$g \geqslant g_{a,b} \text{ with equality at } x_0 \text{ but not everywhere on } (0, \infty) \tag{2.5}$$

("linear minorization"). But now, by Proposition 3(a),

$$Fg > Fg_{a,b} \text{ everywhere on } (0, \infty). \tag{2.6}$$

However, since $Fg = g$ and $Fg_{a,b} = g_{a,b}$, this contradicts (2.5) at x_0. ∎

2.2. *Proof of Theorem 2*

Let $Fg = \lambda g$. We show that if $\lambda \neq 1$ then $g \notin \mathscr{H}$. We distinguish between the cases $\lambda < 1$ and $\lambda > 1$.

$\underline{\lambda < 1}$: Let $f(x) = x^2$. Then by Propositions 1(c–d)

$$([K_g]^n f)(\theta) = \theta^2 + \left(\sum_{m=1}^{n} \lambda^m\right) g(\theta) \qquad (n \geqslant 0), \tag{2.7}$$

where the r.h.s. is convex for all $n \geqslant 0$ by Proposition 4(f′). Define

$$\hat{f}(\theta) = \lim_{n \to \infty} ([K_g]^n f)(\theta) = \theta^2 + \frac{\lambda}{1-\lambda} g(\theta). \tag{2.8}$$

Then $\hat{f}$ is convex and

$$K_g \hat{f} = \hat{f}. \tag{2.9}$$

By the same linear minorization argument as in the proof of Step 2 in Section 2.1, it now follows that $\hat{f}$ must be linear (use Proposition 4(a′)). Thus, $\hat{f}(\theta) = a\theta$ for some $a \in (0, \infty)$. However, this says that $g(\theta) = [(1-\lambda)/\lambda](a\theta - \theta^2)$. So $g(\theta) < 0$ for $\theta > 1/a$ and therefore $g \notin \mathscr{H}$.

$\underline{\lambda > 1}$: By (0.1)(iv), for every $a > 0$ there exists $b = b(a)$ such that

$$g(x) \leqslant ax^2 + b \qquad (x \geqslant 0). \tag{2.10}$$

Applying $[K_g]^n$ to both sides and recalling that $K_g g = Fg = \lambda g$, we obtain with the help Propositions 1(a, c–d) and 4(a′) that

$$\begin{aligned} \lambda^n g(\theta) &= ([K_g]^n g)(\theta) \leqslant ([K_g]^n (ax^2 + b))(\theta) \\ &= (a\theta^2 + b) + a\left(\sum_{m=1}^{n} \lambda^m\right) g(\theta) \qquad (n \geqslant 0). \end{aligned} \tag{2.11}$$

Divide by λ^n and let $n \to \infty$ to get

$$g(\theta) \leqslant a \frac{1}{1-\lambda^{-1}} g(\theta). \tag{2.12}$$

Since a is arbitrary we conclude that $g \equiv 0$ and so $g \notin \mathscr{H}$.

This completes the proof of Theorem 2. In the rest of this section we give an *alternative* proof of Theorem 2, one that is more geometrical and therefore more intuitive. It is based on a domination argument similar to the one used in Section 2.1 to prove Theorem 1.

For $a>0$ and $b, c \in \mathbb{R}$ define *up-parabolas* resp. *down-parabolas* as follows:

$$
\begin{aligned}
g^{\cup}_{a,b}(x) &= a(x-b)^2\, 1_{\{x \geqslant b\}} && (x \in \mathbb{R}) \\
g^{\cap}_{a,b,c}(x) &= a(x-b)(c-x)\, 1_{\{b \leqslant x \leqslant c\}} && (x \in \mathbb{R}).
\end{aligned}
\tag{2.13}
$$

None of these functions is in our class $\mathscr{H}_s$ (recall (2.1)), but they are all *shape invariant* under F:

LEMMA 8. *For all $b, c \in \mathbb{R}$*

$$
\begin{aligned}
&\text{(a)} \quad Fg^{\cup}_{a,b} = \frac{1}{1-a}\, g^{\cup}_{a,b} && (0<a<1) \\
&\text{(b)} \quad Fg^{\cap}_{a,b,c} = \frac{1}{1+a}\, g^{\cap}_{a,b,c} && (a>0).
\end{aligned}
\tag{2.14}
$$

Proof. Immediate from Propositions 1(a–c).[2] Property (a) appeared before as Lemma 5(c) (for $b=0$). Note that the eigenvalues *only* depend on the scale parameter a and not on the shift parameters b, c. ∎

The point that we shall exploit is that both eigenvalues in (2.14) tend to 1 as $a \downarrow 0$.

The key to the alternative proof of Theorem 2 is the following domination lemma analogous to (2.5).

LEMMA 9. *For every $g \in \mathscr{H}$:*

1. ("*up-parabolic majorization*") *For every $a>0$ sufficiently small there exist $b \leqslant 0$ and $x_0 > 0$ such that*

$$
g \leqslant g^{\cup}_{a,b} \text{ with equality at } x_0. \tag{2.15}
$$

[2] The fact that $g^{\cup}_{a,b}$ and $g^{\cap}_{a,b,c}$ are not in $\mathscr{H}_s$ is no problem for F. For the up-parabolas we can define the image under F by the same construction as in (2.1–2.2). For the down-parabolas, on the other hand, we can define the image under F as $\lim_{n\to\infty} Fg_n$, where (g_n) is any decreasing sequence of functions in $\mathscr{H}_s$ that converges to $g^{\cap}_{a,b,c}$ pointwise. Lemma 4 guarantees that the result of this limiting procedure is the same as if we simply restrict the integrations in (1.1–1.2) to the interval $[b, c]$. See also the proof of Proposition 3(c).

2. (*"down-parabolic minorization"*) *For every* $a>0$ *there exist* $c>b\geqslant 0$ *and* $x_0>0$ *such that*

$$g\geqslant g^{\cap}_{a,b,c} \text{ with equality at } x_0. \tag{2.16}$$

Proof. 1. Consider the family $(g^{\cup}_{a,b})_{b\leqslant 0}$ for some $0<a<g(1)$. Since $g(x)=o(x^2)$ $(x\to\infty)$, we obviously have that $g<g^{\cup}_{a,b}$ for b sufficiently small. On the other hand, g intersects $g^{\cup}_{a,b}$ in some point $x\in(0,1)$ for $b=0$. Hence, by continuity, there exists $b\leqslant 0$ for which the claim holds.

2. For every $a>0$ there exist $c>b\geqslant 0$ with $c-b$ sufficiently small such that $g>g^{\cap}_{a,b,c}$. Fix a, b and consider the family $(g^{\cap}_{a,b,c})_{c\geqslant b}$. The top of the down-parabola has height $\frac{1}{4}a(c-b)^2$, which increases quadratically with c. Therefore, since $g(x)=o(x^2)$ $(x\to\infty)$, there exists c large enough so that g intersects $g^{\cap}_{a,b,c}$. By continuity the claim follows. ∎

Using Proposition 3(a) and Lemmas 8–9 we can now give the alternative proof of Theorem 2. Suppose that $g\in\mathscr{H}$ satisfies $Fg=\lambda g$. We want to prove that $\lambda\neq 1$ is not possible. This goes in two steps:

(1) By property 1 in Lemma 9 we have $g\leqslant g^{\cup}_{a,b}$ for *any* $a>0$ sufficiently small and some $b\leqslant 0$. Apply Proposition 3(a) to get $Fg\leqslant Fg^{\cup}_{a,b}$. Together with Lemma 8(a) this yields $\lambda g\leqslant[1/(1-a)]\, g^{\cup}_{a,b}$. But $g(x_0)=g^{\cup}_{a,b}(x_0)>0$. Hence $\lambda\leqslant[1/(1-a)]$. Now let $a\downarrow 0$ to get $\lambda\leqslant 1$.

(2) A similar argument, running via property 2 in Lemma 9 and via Lemma 8(b), gives that $\lambda\geqslant 1/(1+a)$ for *any* $a>0$. Again, let $a\downarrow 0$ to get $\lambda\geqslant 1$.

Combine (1) and (2) to conclude that $\lambda=1$.

2.3. *Proof of Theorem* 3

Fix $g\in\mathscr{H}$ such that $\lim_{x\to\infty} x^{-1}g(x)=a\in(0,\infty)$. Let $g^+\in\mathscr{H}$ be the *concave upper envelope* of g, i.e., the smallest concave function dominating g. Then it follows from Proposition 3(b) that

$$g^+\geqslant Fg^+ \tag{2.17}$$

and hence from Proposition 3(a), after repeatedly applying F to both sides of (2.17), that

$$g^+\geqslant Fg^+\geqslant F^2g^+\geqslant\cdots \tag{2.18}$$

(alternatively, use Proposition 3(f)). Similarly, let $g^-\in\mathscr{H}$ be the *convex lower envelope* of g. Then

$$g^-\leqslant Fg^-\leqslant F^2g^-\leqslant\cdots. \tag{2.19}$$

Moreover, since $g^+ \geqslant g \geqslant g^-$, Proposition 3(a) also gives us the sandwich

$$F^n g^+ \geqslant F^n g \geqslant F^n g^- \qquad (n \geqslant 0). \tag{2.20}$$

Next, define $F^\infty g^+$ and $F^\infty g^-$ to be the pointwise limits of $F^n g^+$ resp. $F^n g^-$ as $n \to \infty$. We claim that both these limits are uniform on compacts in $[0, \infty)$. To see why, note that

$$\lim_{x \to \infty} x^{-1} g^+(x) = \lim_{x \to \infty} x^{-1} g^-(x) = a. \tag{2.21}$$

Since g^+ is Lipschitz at 0 (recall (0.1)(iii)) and since $a < \infty$, (2.21) implies that both g^+ and g^- are *globally* Lipschitz. It therefore follows from Proposition 3(d) that $\{F^n g^+\}_{n \geqslant 0}$ and $\{F^n g^-\}_{n \geqslant 0}$ are uniformly equicontinuous on $[0, \infty)$. Since both sequences are bounded by g^+, the claim follows via the theorem of Arzela-Ascoli.

Next, $F^n g^+$ and $F^n g^-$ in fact converge to $F^\infty g^+$ resp. $F^\infty g^-$ in the metric induced by $\|\cdot\|$, defined in (1.15). Indeed, this immediately follows from the uniform convergence on compacts, together with the fact that g^+ bounds both sequences and $g^+(x) = o(x^2)$ $(x \to \infty)$. It therefore follows from the continuity property of F, as formulated in Proposition 3(c), that pointwise

$$\begin{aligned} F^\infty g^+ &= \lim_{n \to \infty} F(F^n g^+) = F(F^\infty g^+) \\ F^\infty g^- &= \lim_{n \to \infty} F(F^n g^-) = F(F^\infty g^-), \end{aligned} \tag{2.22}$$

i.e., $F^\infty g^+$ and $F^\infty g^-$ are *fixed points* of F. This in turn implies, via Theorem 1, that they must be linear. Hence, in accordance with (2.21),

$$F^\infty g^+ = F^\infty g^- = g_a \tag{2.23}$$

(recall (0.8) for the definition of g_a).

Finally, let $n \to \infty$ in (2.20) and use (2.23) to get that (0.9) in Theorem 3 holds pointwise. To see that (0.9) also holds uniformly on compacts in $[0, \infty)$, simply use that this is true for $g = g^+$ and $g = g^-$ by the above argument, and remember (2.20) and (2.23).

3. PROOF OF THEOREM 5

The proof of Theorem 5(i) is immediate from the argument in Section 2.3. Simply combine the uniform convergence on compacts in $[0, \infty)$ with (2.20–2.21), to conclude that the convergence in (0.9) holds in the τ_∞-topology (defined in (0.15)(ii)).

The proof of Theorems 5(ii–iii) is more delicate. We shall see that to get convergence in the τ_0-topology (defined in (0.15)(i)), it is critical that g stays above a quadratic in a neighborhood of 0. In fact, we shall see that:

(1) if g decays faster than quadratic, then the same is true for $F^n g$ for all n;

(2) if g decays slower than or equal to quadratic, then as n increases $F^n g$ develops a *strictly positive* slope.

3.1. *Preparatory Lemmas*

Lemmas 10–12 below are technical statements about the behavior of F near the boundary at 0 and will be needed for the proof of Theorems 5(ii–iii) in Section 3.2.

For $g \in \mathscr{H}$, define

$$\begin{aligned} s(g) &= \limsup_{x \downarrow 0} x^{-2} g(x) \\ i(g) &= \liminf_{x \downarrow 0} x^{-2} g(x). \end{aligned} \tag{3.1}$$

Define the map T: $[0, \infty) \to [0, \infty]$ by

$$\begin{aligned} T(a) &= \frac{a}{1-a} && \text{if} \quad 0 \leqslant a < 1 \\ &= \infty && \text{if} \quad a \geqslant 1. \end{aligned} \tag{3.2}$$

LEMMA 10. *For all* $g \in \mathscr{H}$

(a) $s(Fg) \leqslant T(s(g))$

(b) $i(Fg) \geqslant T(i(g))$.

In particular, if $s(g) = i(g) = a$ *then* $s(Fg) = i(Fg) = T(a)$.

Proof. Same as Lemma 9 in Part I. The proof relies on Lemma 3 in combination with a sequence of explicit estimates that are given in Part I Sections 2.1 and 3.[3] ∎

[3] The reason why we have assumed $\lim_{x \to \infty} x^{-2} g(x) = 0$ in (0.1)(iv) is not just that the transformation F can under this assumption be iterated (recall Lemma 5 in Section 1.2), but also that the left and the right boundary are "decoupled," i.e., there is no interference between the qualitative properties of Fg near 0 and near ∞. Consequently, all the analysis that was done in Part I for the boundary behavior of the transformation on $[0, 1]$ near 0 carries over to the present situation.

LEMMA 11. *If* $g(x) \sim ax^2$ $(x \downarrow 0,\ 1 < a < \infty)$, *then* $(Fg)(x) \sim C_a x^{1+(1/a)}$ $(x \downarrow 0,\ C_a > 0)$.

Proof. See Part I Propositions 6–7 and Section 2.3. ∎

LEMMA 12. *If* $x \to x/g(x)$ *is integrable at* 0, *then*

$$\lim_{\theta \downarrow 0} \theta^{-1}(Fg)(\theta) = \int_0^\infty dx \exp\left[-\int_0^x dy \frac{y}{g(y)} \right]. \tag{3.3}$$

Proof. Let $N(\theta)$ and $D(\theta)$ denote the numerator resp. denominator of (1.2). The proof amounts to showing that:

(i) $\lim_{\theta \downarrow 0} \theta D(\theta) = 1$ when $x \to x/g(x)$ is integrable at 0;

(ii) $\lim_{\theta \downarrow 0} N(\theta) = \text{r.h.s. } (3.3)$.

Property (i) is the same as Part I Lemma 6 and uses the estimates in Part I Lemma 5. Property (ii) is the same as Part I Equation (2.29), and follows from (1.1) via the observation that

$$\lim_{\theta \downarrow 0} g(x)\, \mu_\theta^g(x) = \exp\left[-\int_0^x dy \frac{y}{g(y)} \right] \qquad (x > 0), \tag{3.4}$$

where the convergence is monotone for $\theta < x$. ∎

3.2. *Proof of Theorem* 5(ii–iii)

The key step in the proof is the following lemma. Define for $n \geq 1$

$$c_n(g) = \lim_{\theta \downarrow 0} \theta^{-1}(F^n g)(\theta). \tag{3.5}$$

The limit in (3.5) exists by Lemma 6(b), because $F^{n-1}g$ is Lipschitz at 0.

LEMMA 13. *Suppose that* $\lim_{x \to \infty} x^{-1} g(x) = a \in (0, \infty)$. *Let* g^+, g^- *be the concave upper resp. convex lower envelope of* g (*as in Section* 2.3).

(a) *If* $\liminf_{x \downarrow 0} x^{-2} g(x) > 0$, *then*

$$\lim_{n \to \infty} c_n(g^+) = \lim_{n \to \infty} c_n(g^-) = a. \tag{3.6}$$

(b) *If* $\limsup_{x \downarrow 0} x^{-2} g(x) = 0$, *then*

$$c_n(g^+) = c_n(g^-) = 0 \qquad \textit{for all} \quad n \geq 1. \tag{3.7}$$

Proof. Lemma 13(b) is immediate from (3.2) and Lemma 10(a). Indeed, if $s(g)=0$ then $s(F^n g)=0$ for all $n \geqslant 1$. Hence $c_n(g)=0$ for all $n \geqslant 1$ (see (3.1) and (3.5)).

The proof of Lemma 13(a) comes in three steps.

1. Use (3.2) and Lemma 10(b) to see that if $i(g)>0$, then there exists $0 \leqslant n_0 < \infty$ for which $i(F^{n_0-1}g) < 1 < i(F^{n_0}g)$.

2. Combining Lemma 11 with $i(F^{n_0}g)>1$ (and recalling Proposition 3(a)) we get that $x \to x/(F^{n_0+1}g)(x)$ is integrable at 0, i.e., the decay at 0 is slower than quadratic.

3. From Lemma 12 we therefore have the following representation (see (3.3) and (3.5)):

$$c_{n+1}(g) = \int_0^\infty dx \exp\left[-\int_0^x dy \frac{y}{(F^n g)(y)}\right] \qquad (n \geqslant n_0+1). \tag{3.8}$$

Now pick $g=g^+, g^-$ and use that $F^n g^+$ and $F^n g^-$ both tend to g_a pointwise and monotonically as $n \to \infty$ (see Section 2.3). Then, by monotone convergence,

$$\lim_{n\to\infty} c_{n+1}(g^+) = \lim_{n\to\infty} c_{n+1}(g^-) = \int_0^\infty dx \exp\left[-\int_0^x dy \frac{y}{g_a(y)}\right] = a. \quad \blacksquare \tag{3.9}$$

We can now complete the proof of Theorem 5(ii–iii). Theorem 5(iii) is immediate from Lemma 13(b), because $c_n(g)=0$ implies that $\|F^n g - g_a\|_{(0,N)} \geqslant a$ for any $N>0$ (recall (0.14)). To get Theorem 5(ii), use Proposition 3(f) to see that $F^n g^+$ is concave and $F^n g^-$ is convex for all n. Therefore (3.5–3.6) imply that $(F^n g^+)'$ and $(F^n g^-)'$ converge to a uniformly on $(0,\infty)$. Together with (2.20) this gives that $\lim_{n\to\infty} \|F^n g - g_a\|_{(0,N)} = 0$ for all $N<\infty$ (recall (0.14)).

4. PROOF OF THEOREMS 4 AND 6

4.1. *Heuristic Background*

We start with a heuristic discussion of the result in Theorem 4, since this will help us to devise a strategy for the proof. Our task is to investigate $(F^n g)(\theta)$ for fixed $\theta \in (0,\infty)$ and $n \to \infty$. In view of Proposition 2(d), this means that we need to collect information about the kernel $K_g^{(n)}(\theta, dy)$ defined in (1.13). Along the way it will be necessary to place some regularity assumptions on g. It turns out that the right notion is *regular* *variation.* For that reason we shall make the assumption stated in Theorem 4:

$$g(x) \sim x^\alpha L(x),\ \alpha \in (0, 2)\backslash\{1\} \text{ and } L(x) \text{ slowly varying at infinity.} \tag{4.1}$$

As a guide we have the situation in the special case $g = g_a$. In this case $K_{g_a}^{(n)} = [K_{g_a}]^n$ (recall that $g_a(x) = ax$ is a fixed point of F). Moreover, K_{g_a} has the so-called "branching property":

$$K_{g_a}(\theta_1 + \theta_2, \cdot) = K_{g_a}(\theta_1, \cdot) * K_{g_a}(\theta_2, \cdot), \tag{4.2}$$

where $*$ denotes convolution of measures. Using this property in connection with moment formulas, it is possible to show (with the help of Laplace transform techniques [DG4]) that

$$\begin{gathered}\lim_{n\to\infty} K_{g_a}^{(n)}(\theta, [\varepsilon, \infty)) = 0 \\ \lim_{n\to\infty} \frac{K_{g_a}^{(n)}(\theta, nI \cap [\varepsilon, \infty))}{K_{g_a}^{(n)}(\theta, [\varepsilon, \infty))} = \frac{1}{a}\int_I dy\, e^{-y/a}\end{gathered} \tag{4.3}$$

for any $\theta > 0$, any interval $I \subset (0, \infty)$ and any $\varepsilon > 0$. This means that the distribution $K_{g_a}^{(n)}(\theta, \cdot)$ has the following shape: "Close to 0 sits almost all of the mass, while almost all of the remaining mass is distributed on scale n according to an exponential law with parameter $1/a$."

In the situation where g satisfies (4.1) we expect that (4.3) changes as follows:

(I) scale n becomes scale $e_n = e_n(g)$ defined by (0.10);

(II) the law of the mass on scale e_n remains exponential (with some g-dependent parameter).

Assuming (I) and (II) we can use Proposition 2 to determine $F^n g$. Namely, let m_n be the mass on scale e_n and $1 - m_n$ the mass around 0. Let μ be the mass distribution on scale e_n. Let $\hat{e}_n$ be the multiple of e_n such that $\mu(\hat{e}_n \cdot)$ is exponential with parameter 1. Then, as $n \to \infty$, the three relations in Propositions 2(b–d) give us, respectively,

$$\begin{gathered} m_n \hat{e}_n 1! \sim \theta \\ m_n \hat{e}_n^2 2! \sim n(F^n g)(\theta) \\ m_n g(\hat{e}_n)\, \alpha! \sim (F^n g)(\theta). \end{gathered} \tag{4.4}$$

(In the second relation we have dropped a term θ^2 because the r.h.s. tends to ∞. In the third relation we have used that $g(\hat{e}_n x) \sim x^\alpha g(\hat{e}_n)$. The latter follows from (4.1) via [BGT] Theorem 1.5.2.) From (4.4) we can solve

$$\frac{g(\hat{e}_n)}{\hat{e}_n^2} \sim \frac{2!}{\alpha!}\frac{1}{n}, \qquad \frac{n}{\hat{e}_n}(F^n g)(\theta) \sim \frac{2!}{1!}\theta. \tag{4.5}$$

Now put

$$\hat{e}_n = \left(\frac{\alpha!}{2!}\right)^{1/(2-\alpha)} e_n. \tag{4.6}$$

Then (4.5) turns into

$$\frac{g(e_n)}{e_n^2} \sim \frac{1}{n}, \qquad \frac{n}{e_n}(F^n g)(\theta) \sim \left(\frac{2!}{1!}\right)\left(\frac{\alpha!}{2!}\right)^{1/(2-\alpha)} \theta, \tag{4.7}$$

which precisely corroborates the conjecture (0.13).

Unfortunately, we are not able to make (I–II) and (4.4) rigorous. In order to derive a version of (4.4) that will prove Theorem 4, we shall have to work with bounds from above and below. These bounds will use the properties established in Propositions 3–4 and the explicitly solvable cases $g(x) = ax(b+x)$ and $g(x) = ax(b-x)^+$, i.e., the up- and down-parabolas introduced in Section 2.2.

In order for our considerations to make sense, we need to show that the sequence $(e_n)_{n \geqslant 1}$ is asymptotically uniquely determined by the behavior of $g(x)$ as $x \to \infty$ (recall (0.10)). This is formulated in the following lemma.

LEMMA 14. (a) *Let $g \in \mathscr{H}$ satisfy* (4.1). *Let $(e'_n)_{n \geqslant 1}$ and $(e''_n)_{n \geqslant 1}$ be any two solutions of the equation*

$$\frac{1}{n} = \frac{g(e_n)}{e_n^2} \qquad (n \geqslant 1). \tag{4.8}$$

Then

$$\lim_{n \to \infty} \frac{e'_n}{e''_n} = 1. \tag{4.9}$$

(b) *Let $g_1, g_2 \in \mathscr{H}$ satisfy* (4.1) *and*

$$\lim_{x \to \infty} \frac{g_1(x)}{g_2(x)} = K \in (0, \infty). \tag{4.10}$$

Let $(e_{n,1})_{n\geqslant 1}$ *and* $(e_{n,2})_{n\geqslant 1}$ *be any two solutions of the equations*

$$\frac{1}{n}=\frac{g_i(e_{n,i})}{e_{n,i}^2} \qquad (n\geqslant 1;\, i=1,2). \tag{4.11}$$

Then

$$\lim_{n\to\infty}\frac{e_{n,1}}{e_{n,2}}=K^{1/(2-\alpha)}. \tag{4.12}$$

Proof. (a) Define $h(x)=x^2/g(x)$. Then (4.8) reads $n=h(e_n)$, i.e., $n\to e_n$ is the inverse of $x\to h(x)$. Now, the function h is regularly varying with index $2-\alpha$. It therefore follows from [BGT] Theorem 1.5.12 that e_n is determined uniquely up to asymptotic equivalence (and is regularly varying with index $1/(2-\alpha)$).

(b) This is a direct consequence of the conjugacy properties stated in [BGT] Propositions 1.5.14–1.5.15. ∎

In Sections 4.2–4.5 we prove upper and lower bounds for the two cases $\alpha\in(0,1)$ and $\alpha\in(1,2)$. The guiding idea is that, for large n, $F^n g$ is determined by what g looks like on scale $e_n=e_n(g)$. Therefore our bounds will be successful if we manage to approximate g well in the neighborhood of e_n. The approximation is done by parabolas, and therefore we loose something in the constants.

4.2. *Lower Bound for* $\alpha\in(0,1)$

The derivation of the lower bound for $\alpha\in(0,1)$ proceeds in two steps: (1) derivation under the assumption that $x\to g(x)/x$ is decreasing; (2) removal of this restriction.

Step 1. $x\to g(x)/x$ decreasing.

Recall from Proposition 3(a) that $g_1\leqslant g_2$ implies $Fg_1\leqslant Fg_2$. In what follows we shall bound g from below by a *down-parabola*, chosen in such a way that it approximates g well in the neighborhood of $e_n=e_n(g)$. The iterates under F of this down-parabola are easy to compute and provide a lower bound for the iterates under F of g. For the method to work the constants have to be chosen carefully.

For $\delta\in(0,1)$, define

$$g_{n,\delta}^{\frown}(x)=\frac{\delta(1-\alpha)}{n}\,x\left(\frac{2-\alpha}{1-\alpha}\,e_n-x\right)^{+}. \tag{4.13}$$

Note that $g_{n,\delta}^{\frown}(e_n)=\delta e_n^2/n=\delta g(e_n)$ by (0.10).

LEMMA 15. *For every* $\delta \in (0, 1)$ *there exists* $n_0 = n_0(\delta)$ *such that*

$$g \geqslant g^{\cap}_{n,\delta} \qquad \textit{for all} \quad n \geqslant n_0. \tag{4.14}$$

Before we prove this lemma, we first complete the argument of the lower bound. Applying Proposition 3(a) and Lemma 8(b), we get

$$F^n g \geqslant F^n g^{\cap}_{n,\delta} = \frac{1}{1 + n[\delta(1-\alpha)/n]}\, g^{\cap}_{n,\delta} = \frac{1}{1+\delta(1-\alpha)}\, g^{\cap}_{n,\delta} \qquad (n \geqslant n_0). \tag{4.15}$$

Hence

$$\begin{aligned} \liminf_{n \to \infty} \frac{1}{\theta} \frac{n}{e_n} (F^n g)(\theta) &\geqslant \frac{1}{1+\delta(1-\alpha)} \liminf_{n \to \infty} \frac{1}{\theta} \frac{n}{e_n}\, g^{\cap}_{n,\delta}(\theta) \\ &= \frac{\delta(1-\alpha)}{1+\delta(1-\alpha)} \frac{2-\alpha}{1-\alpha}. \end{aligned} \tag{4.16}$$

Now let $\delta \uparrow 1$ to obtain

$$\liminf_{n \to \infty} \frac{1}{\theta} \frac{n}{e_n} (F^n g)(\theta) \geqslant 1. \tag{4.17}$$

This is the lower bound in (0.11–0.12). Note that (4.16–4.17) are in fact valid for $\theta \in (0, o(e_n))$, so also the corresponding bound in (0.16) follows.

Proof of Lemma 15. Recall that

$$\frac{1}{n} = \frac{g(e_n)}{e_n^2}. \tag{4.18}$$

With this relation we can write (4.14) as the condition

$$\frac{g(x)}{g(e_n)} \geqslant \frac{\delta(1-\alpha)}{e_n^2}\, x \left(\frac{2-\alpha}{1-\alpha}\, e_n - x\right)^+ \qquad (x \geqslant 0). \tag{4.19}$$

Put $x = y e_n$. Then (4.19) becomes

$$\frac{g(y e_n)}{g(e_n)} \geqslant \delta(1-\alpha)\, y \left(\frac{2-\alpha}{1-\alpha} - y\right) \qquad \left(y \in \left[0, \frac{2-\alpha}{1-\alpha}\right]\right). \tag{4.20}$$

Since g is regularly varying with index α and $e_n \to \infty$, we have by [BGT] Theorem 1.5.2 that

$$\lim_{n \to \infty} \frac{g(y e_n)}{g(e_n)} = y^{\alpha} \text{ uniformly on compacts in } [0, \infty). \tag{4.21}$$

Furthermore, the following inequality holds:

$$y^{\alpha} \geqslant (1-\alpha)\, y \left(\frac{2-\alpha}{1-\alpha} - y\right) \qquad (y \geqslant 0). \tag{4.22}$$

To check this, note that (4.22) is equivalent to $(1-\alpha)(y-1) \geqslant 1 - y^{-(1-\alpha)} \geqslant 0$. This is correct for $y=1$, and we are done once we establish that $f(y) = (1-\alpha)(y-1) - 1 + y^{-(1-\alpha)}$ attains a minimum at $y=1$. But $f'(y) = (1-\alpha)(1 - y^{-(2-\alpha)})$, which changes sign at $y=1$.

Combining (4.21–4.22) and using that $\delta < 1$, we now know that (4.20) holds uniformly on compacts in $(0, \infty)$ for n sufficiently large. Thus, it suffices to check (4.20) in a neighborhood of 0. At this point the assumption that $x \to g(x)/x$ is decreasing comes in. Namely, pick $\varepsilon > 0$. Then, for all $y \in [0, \varepsilon]$, the l.h.s. of (4.20) is larger than $yg(\varepsilon e_n)/\varepsilon g(e_n)$. But $g(\varepsilon e_n)/\varepsilon g(e_n) \sim \varepsilon^{\alpha-1}$ as $\varepsilon \downarrow 0$ by (4.21). Since $\alpha < 1$, the inequality in (4.20) therefore also holds on $[0, \varepsilon]$ for ε sufficiently small and n sufficiently large. ∎

Step 2. Removal of $x \to g(x)/x$ decreasing.

The idea is to approximate g by a completely monotone function. We have the following fact from [BGT] Theorem 1.8.3: For every $g \in \mathscr{H}$ satisfying (4.1) there exists $\hat{g} \in \mathscr{H}$ such that

$$\begin{aligned} &\text{(i)} \quad \lim_{x\to\infty} \frac{\hat{g}(x)}{g(x)} = 1 \\ &\text{(ii)} \quad \hat{g} \text{ is completely monotone.} \end{aligned} \tag{4.23}$$

(For these properties to be true it is important that α in (4.1) is not integer.) Since we are dealing with the case $0 < \alpha < 1$, we must have

$$\hat{g} \geqslant 0, \hat{g}' \geqslant 0, \hat{g}'' \leqslant 0. \tag{4.24}$$

Using these facts, we proceed as follows. For every $\varepsilon > 0$ there exists $x_0 = x_0(\varepsilon)$ such that

$$g(x) \geqslant (1-\varepsilon)\, \hat{g}(x) \qquad \text{for all} \quad x \geqslant x_0. \tag{4.25}$$

Moreover, for every $\delta > 0$ there exists $\delta' = \delta'(\delta, x_0) > 0$ such that

$$g(x) \geqslant \delta'(x-\delta)^{+} \qquad \text{for all} \quad 0 \leqslant x < x_0. \tag{4.26}$$

(Here we shift by δ to allow for g with $\lim_{x\downarrow 0} g(x)/x = 0$.) We now define the function

$$f(x) = \min\{(1-\varepsilon)\, \hat{g}(x), \delta'(x-\delta)^{+}\} \qquad (x \geqslant 0). \tag{4.27}$$

This function satisfies

$$\begin{aligned} &\text{(i)} \quad g \geqslant f \\ &\text{(ii)} \quad x \to \frac{f(x)}{x-\delta} \qquad \text{decreasing on } (\delta, \infty) \\ &\text{(iii)} \quad \lim_{x\to\infty} \frac{f(x)}{(1-\varepsilon)\,\hat{g}(x)} = 1 \end{aligned} \tag{4.28}$$

((i) follows from (4.25–4.26); (ii) holds because $\hat{g}$ is concave; (iii) holds because $\hat{g}$ is sublinear). Now, by Proposition 3(a) and (4.28)(i) we have that

$$F^n g \geqslant F^n f. \tag{4.29}$$

Moreover, by (4.28)(ii) we can apply the result from Step 1 to f: $[\delta, \infty) \to [0, \infty)$ and conclude that, as in (4.17),

$$\liminf_{n\to\infty} \frac{1}{\theta} \frac{n}{e_n(f)} (F^n f)(\theta) \geqslant 1 \qquad (\theta \geqslant \delta). \tag{4.30}$$

Since δ can be picked arbitrarily small, this limit in fact holds for all $\theta > 0$. Finally, by Lemma 14, (4.28)(iii) and (4.23)(i),

$$\begin{aligned} &\lim_{n\to\infty} \frac{e_n(f)}{e_n((1-\varepsilon)\,\hat{g})} = 1 \\ &\lim_{n\to\infty} \frac{e_n((1-\varepsilon)\,\hat{g})}{e_n(\hat{g})} = (1-\varepsilon)^{1/(2-\alpha)} \\ &\lim_{n\to\infty} \frac{e_n(\hat{g})}{e_n(g)} = 1. \end{aligned} \tag{4.31}$$

Combining (4.29–4.31) and letting $\varepsilon \downarrow 0$, we again arrive at (4.17). This completes the proof of the lower bound for $\alpha \in (0, 1)$.

4.3. *Upper Bound for* $\alpha \in (1, 2)$

The strategy here is similar to the one followed in Section 4.2. Again we proceed in two steps: (1) derivation under the assumption that $x \to g(x)/x$ is increasing; (2) removal of this restriction.

Step 1. $x \to g(x)/x$ increasing.

This time we use an *up-parabola* to bound g from above. This up-parabola is chosen in such a way that it approximates g well in the neighborhood of $e_n = e_n(g)$. Indeed, for $\delta \in (1, \infty)$, define

$$g^{\cup}_{n,\delta}(x) = \frac{\delta(\alpha-1)}{n}\, x\left(\frac{2-\alpha}{\alpha-1}\, e_n + x\right). \tag{4.32}$$

Note that $g^{\cup}_{n,\delta}(e_n) = \delta e_n^2/n = \delta g(e_n)$ by (0.10).

LEMMA 16. *For every $\delta \in (1, \infty)$ there exists $n_0 = n_0(\delta)$ such that*

$$g \leqslant g^{\cup}_{n,\delta} \qquad \textit{for all} \quad n \geqslant n_0. \tag{4.33}$$

Before we prove this lemma, we first complete the argument of the upper bound. The analogues of (4.15–4.16), based on Proposition 3(a) and Lemma 8(a), read

$$F^n g \leqslant \frac{1}{1 - n[\delta(\alpha-1)/n]}\, g^{\cup}_{n,\delta} = \frac{1}{1-\delta(\alpha-1)}\, g^{\cup}_{n,\delta} \tag{4.34}$$

(pick $1 < \delta < 1/(\alpha-1)$) respectively

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{\theta}\frac{n}{e_n}\,(F^n g)(\theta) &\leqslant \frac{1}{1-\delta(\alpha-1)} \limsup_{n\to\infty} \frac{1}{\theta}\frac{n}{e_n}\, g^{\cup}_{n,\delta}(\theta) \\ &= \frac{\delta(\alpha-1)}{1-\delta(\alpha-1)}\,\frac{2-\alpha}{\alpha-1}. \end{aligned} \tag{4.35}$$

Let $\delta \downarrow 1$ to obtain

$$\limsup_{n\to\infty} \frac{1}{\theta}\frac{n}{e_n}\,(F^n g)(\theta) \leqslant 1. \tag{4.36}$$

This is the upper bound in (0.11–0.12). Note that (4.35–4.36) are in fact valid for $\theta \in (0, o(e_n))$, so also the corresponding bound in (0.16) follows.

Proof of Lemma 16. Use (4.18) to write (4.33) as the condition

$$\frac{g(ye_n)}{g(e_n)} \leqslant \delta(\alpha-1)\, y\left(\frac{2-\alpha}{\alpha-1} + y\right) \qquad (y \geqslant 0). \tag{4.37}$$

Using the fact that g is regularly varying with index α, we again have (4.21):

$$\lim_{n\to\infty} \frac{g(ye_n)}{g(e_n)} = y^{\alpha} \text{ uniformly on compacts in } [0, \infty). \tag{4.38}$$

But (compare with (4.22))

$$(\alpha-1)\, y\left(\frac{2-\alpha}{\alpha-1}+y\right)\geqslant y^{\alpha} \qquad (y\geqslant 0). \tag{4.39}$$

(As before, this holds for $y=1$ and $f(y)=(\alpha-1)(y-1)-y^{\alpha-1}+1$ takes its minimum at $y=1$.) Combining (4.38–4.39), we see that (4.37) holds uniformly on compacts in $(0, \infty)$ for n sufficiently large. To see that it also holds in a neighborhood of 0, pick $\varepsilon>0$ and use the assumption that $x\to g(x)/x$ is increasing. Then, for all $y\in[0, \varepsilon]$, the l.h.s. of (4.37) is smaller than $yg(\varepsilon e_n)/\varepsilon g(e_n)$. But $g(\varepsilon e_n)/\varepsilon g(e_n)\sim\varepsilon^{\alpha-1}$ as $\varepsilon\downarrow 0$ by (4.28). Since $\alpha>1$, we conclude that the inequality in (4.37) also holds on $[0, \varepsilon]$ for ε sufficiently small and n sufficiently large. Thus, it remains to check (4.37) for y in a neighborhood of ∞. But this follows easily from the observation that the r.h.s. grows like y^2 and the l.h.s. like y^{α}. (See also [BGT] Theorem 1.5.6.) ∎

Step 2. Removal of $x\to g(x)/x$ increasing.

This is completely analogous to Step 2 in Section 4.2 and we leave the details to the reader.

4.4. *Lower Bound for* $\alpha\in(1, 2)$

In Section 4.3 we were able to bound g from above by an up-parabola $g_n^{\cup}$. This gave $F^n g\leqslant F^n g_n^{\cup}$, and the r.h.s. could be computed explicitly. The approach in this section, however, must be different. The reason is that we cannot bound g from below by an up-parabola, simply because $\lim_{x\to\infty} g(x)/x^2=0$. What we shall do is *by iteration* construct a family $(h_n)_{n\geqslant 1}$ of functions such that $F^n g\geqslant h_{n+1}$. Each h_{n+1} is an up-parabola on “most” of the space relevant for approximating $F^n g$ and a straight line on the “rest” of the space.

The calculation requires several new elements compared to what was done in Section 4.3. In particular, we need an estimate showing that the contribution of the piece where h_{n+1} is a straight line (rather than an up-parabola) is asymptotically negligible. This in essence means that $\{h_n\}_{n\geqslant 1}$ compare well with the iterates of an up-parabola. The latter can be calculated explicitly and yield the desired bound.

The proof proceeds in two steps: (1) derivation under the assumption that g is convex, g' exists and is regularly varying (with index $\alpha-1$), $g'(0)>0$, and $x\to g(x)/x^2$ is strictly decreasing; (2) removal of these restrictions.

Step 1. Proof under the above restrictions on g.

The main ingredients for our construction of the approximating family are the following.

(i) Since $x \to g(x)/x^2$ is strictly decreasing, the sequence (e_n) is uniquely defined and strictly increasing to infinity (see (4.8)). For $n \geqslant 1$, let $A_n, B_n \in (0, \infty)$ and define $g_n^{\cup}$ by

$$g_n^{\cup}(x) = \frac{A_n}{n} x(B_n e_n + x). \tag{4.40}$$

These are up-parabolas with slope $A_n B_n(e_n/n)$ at $x=0$. Recall that e_n/n is the scale factor we want to achieve. We shall later specify how to choose A_n and B_n in an optimal way for our purpose of approximating $F^n g$.

(ii) Let l_n be the linear function

$$l_n(x) = \hat{a}_n + \hat{b}_n x \tag{4.41}$$

that touches the function g in $x = ce_n$, i.e., $\hat{a}_n = g(ce_n) - ce_n g'(ce_n)$ and $\hat{b}_n = g'(ce_n)$. Here $c \in (0, \infty)$ is a parameter that we shall choose at the end of the proof in order to optimize the bounds. Since g is convex we know that

$$g \geqslant l_n. \tag{4.42}$$

Furthermore, since g' is regularly varying we have $\lim_{n \to \infty} ce_n g'(ce_n)/g(ce_n) = \alpha$ (see [BGT] Theorem 1.5.11), so for $n \to \infty$

$$\begin{aligned} \hat{a}_n &\sim -(\alpha - 1)\, c^{\alpha} g(e_n) \\ \hat{b}_n &\sim \alpha c^{\alpha - 1} \frac{g(e_n)}{e_n} \end{aligned} \tag{4.43}$$

(use also (4.21)).

(iii) Let $0 < e_n^- < e_n^+ < \infty$ be the two solutions of

$$l_n(x) = g_n^{\cup}(x). \tag{4.44}$$

The existence of two such solutions imposes a restriction on the possible choices for A_n and B_n, as will be displayed in Lemma 17 below. For admissible choices we can define

$$h_n(x) = \begin{cases} g_n^{\cup}(x) & \text{for } \; x \leqslant e_n^- \\ l_n(x) & \text{for } \; x \geqslant e_n^- . \end{cases} \tag{4.45}$$

Note that, because $g'(0) > 0$ and $\alpha > 1$, we can choose $A_1 B_1$ small enough so that $g \geqslant h_1$. In order to make sure that h_n is well defined we need the following simple lemma.

LEMMA 17. *Let*

$$a_n = \hat{a}_n \frac{1}{g(e_n)}$$
$$b_n = \hat{b}_n \frac{e_n}{g(e_n)}. \tag{4.46}$$

If A_n, B_n satisfy

$$b_n - A_n B_n > \sqrt{D_n}$$
$$D_n = (b_n - A_n B_n)^2 + 4a_n A_n > 0, \tag{4.47}$$

then e_n^-, e_n^+ exist and are strictly positive.

Proof. Equation (4.44) written out in standard quadratic form reads

$$\left(\frac{A_n}{n}\right) x^2 - \left(\hat{b}_n - \frac{A_n B_n e_n}{n}\right) x - \hat{a}_n = 0. \tag{4.48}$$

Now use (4.46) and the defining relation for e_n (i.e., $g(e_n)/e_n^2 = 1/n$), to rewrite (4.48) as

$$A_n \left(\frac{x}{e_n}\right)^2 - (b_n - A_n B_n)\left(\frac{x}{e_n}\right) - a_n = 0. \tag{4.49}$$

This gives precisely the conditions on A_n, B_n formulated in (4.47). ∎

The key lemma of this section reads:

LEMMA 18. *For every $\varepsilon > 0$ there exist $\delta(\varepsilon) > 0$ and sequences (A_n), (B_n) satisfying* (4.47) *such that*

$$\liminf_{n \to \infty} A_n B_n \geqslant K_1(\alpha) - \varepsilon, \text{ with } K_1(\alpha) = (2-\alpha)^{1/(2-\alpha)},$$
$$\liminf_{n \to \infty} \frac{e_{n+1}^-}{e_n} \geqslant \delta(\varepsilon) \tag{4.50}$$

and such that the corresponding sequence of functions (h_n) has the property

$$g \geqslant h_1$$
$$Fh_n \geqslant h_{n+1} \qquad \text{on } [0, e_{n+1}^-] \qquad (n \geqslant 1). \tag{4.51}$$

With this lemma we can complete the proof of the lower bound as follows. By (4.51), $g \geqslant h_1$ and hence $Fg \geqslant Fh_1$ (recall Proposition 3(a)). By (4.51), we also know that $Fh_1 \geqslant h_2$ on $[0, e_2^-]$, so $Fg \geqslant h_2$ on $[0, e_2^-]$. But, g being convex, we also know that $Fg \geqslant g \geqslant l_2$ (recall Proposition 3(b) and (4.42)). Therefore, using that $h_2 = l_2$ on $[e_2^+, \infty)$, we get that $Fg \geqslant h_2$. Since convexity is preserved under F (recall Proposition 3(f)), we can repeat the above argument and so we get by induction

$$F^n g \geqslant h_{n+1} \qquad (n \geqslant 0). \tag{4.52}$$

Now, by definition,

$$h_{n+1}(x) = \frac{A_{n+1}}{n+1} x(B_{n+1} e_{n+1} + x) \qquad \text{for} \quad x \leqslant e_{n+1}^- \qquad (n \geqslant 0) \tag{4.53}$$

and hence we obtain (recall that $e_{n+1} \geqslant e_n$)

$$\begin{aligned} \frac{1}{\theta} \frac{n}{e_n} (F^n g)(\theta) &\geqslant \frac{n}{e_n} \frac{A_{n+1}}{n+1} (B_{n+1} e_{n+1} + \theta) \\ &\geqslant \frac{n}{n+1} A_{n+1} B_{n+1} \qquad \text{for} \quad \theta \leqslant e_{n+1}^- \qquad (n \geqslant 0). \end{aligned} \tag{4.54}$$

According to Lemma 18 the sequences (A_n), (B_n) can be chosen such that (4.50) holds and so we get

$$\liminf_{n \to \infty} \frac{1}{\theta} \frac{n}{e_n} (F^n g)(\theta) \geqslant K_1(\alpha) - \varepsilon, \tag{4.55}$$

the limit being uniform for $\theta \in (0, \delta e_n)$. Let $\varepsilon \downarrow 0$ to get the lower bound in (0.11–0.12) and the corresponding bound in (0.16).

We are left with the task to prove Lemma 18.

Proof of Lemma 18. The key point is to obtain information about Fh_n on the interval $[0, e_{n+1}^-]$. To do so we shall exploit properties of the kernel K_g defined in (1.12), in particular its action on convex functions and the formula for its action on quadratic functions. This way we shall be able to reduce the necessary estimates to ones involving the kernel $K_{g_n^\cup}$, which we can calculate explicitly.

In order to compare Fh_n and h_{n+1}, we shall need another function f_n defined as follows:

$$f_n(x) = \begin{cases} g_n^\cup(x) & \text{for} \quad x \leqslant e_n^+ \\ l_n(x) & \text{for} \quad x \geqslant e_n^+. \end{cases} \tag{4.56}$$

Note from (4.44–4.45) that $h_n \geqslant f_n$ and hence

$$Fh_n \geqslant Ff_n. \tag{4.57}$$

Next, the function Ff_n can be expressed as follows using the kernel K_g:

$$Ff_n = K_{f_n}(f_n) = K_{g_n^\cup}(g_n^\cup) - \Delta_n - \tilde{\Delta}_n, \tag{4.58}$$

where we define

$$\begin{aligned} \Delta_n &= K_{f_n}(g_n^\cup - f_n) \\ \tilde{\Delta}_n &= K_{g_n^\cup}(g_n^\cup) - K_{f_n}(g_n^\cup). \end{aligned} \tag{4.59}$$

Now, by Lemma 8(a) the first term in (4.58) is explicitly known to be

$$K_{g_n^\cup}(g_n^\cup) = \left(1 - \frac{A_n}{n}\right)^{-1} g_n^\cup. \tag{4.60}$$

On the other hand, by Propositions 1(b–c) we have

$$\begin{aligned} \tilde{\Delta}_n &= \frac{A_n}{n}(Fg_n^\cup - Ff_n) \\ &= \frac{A_n}{n}[K_{g_n^\cup}(g_n^\cup) - K_{f_n}(f_n)] \\ &= \frac{A_n}{n}(\Delta_n + \tilde{\Delta}_n) \end{aligned} \tag{4.61}$$

and hence

$$\tilde{\Delta}_n = \frac{A_n}{n}\left(1 - \frac{A_n}{n}\right)^{-1} \Delta_n. \tag{4.62}$$

Combining (4.57–4.58), (4.60), and (4.62) we arrive at

$$Fh_n \geqslant \left(1 - \frac{A_n}{n}\right)^{-1} (g_n^\cup - \Delta_n). \tag{4.63}$$

Thus we have, up to the term Δ_n, an estimate where the r.h.s. can be easily compared with h_{n+1}, since both $g_n^\cup$ and h_{n+1} are explicitly known functions. Hence what remains is to show that Δ_n is small compared to $g_n^\cup$ for n large. This is formulated in the following lemma, which will be proved later.

LEMMA 19. *Suppose that* $\lim_{n\to\infty} A_n = A$ *and* $\lim_{n\to\infty} B_n = B$ *with* $A, B \in (0, \infty)$. *Then there exists a sequence* (δ_n) *such that*

$$\begin{gathered} \Delta_n \leqslant \delta_n g_n^{\cup} \qquad \text{on } [0, e_{n+1}^{-}] \qquad (n \geqslant 1) \\ \lim_{n\to\infty} n\delta_n = 0. \end{gathered} \tag{4.64}$$

We continue the proof of Lemma 18. Since $h_{n+1} = g_{n+1}^{\cup}$ on $[0, e_{n+1}^{-}]$, it follows from (4.63–4.64) that the following condition on the sequences (A_n), (B_n) implies the second statement in (4.51):

$$\begin{aligned} &(1-\delta_n)\left(1-\frac{A_n}{n}\right)^{-1}\frac{A_n}{n}\,x(B_n e_n + x) \\ &\quad\geqslant \frac{A_{n+1}}{n+1}\,x(B_{n+1}e_{n+1}+x) \qquad \text{for} \quad x \leqslant e_{n+1}^{-}. \end{aligned} \tag{4.65}$$

If $B_{n+1}e_{n+1} \geqslant B_n e_n$, then this condition is sharpest as $x \downarrow 0$ and so it suffices to have (recall that $e_{n+1} \geqslant e_n$)

$$\begin{aligned} &A_{n+1}B_{n+1} \leqslant (1-\delta_n)\frac{e_n}{e_{n+1}}\frac{n+1}{n}\left(1-\frac{A_n}{n}\right)^{-1} A_n B_n \\ &B_{n+1} \geqslant B_n. \end{aligned} \tag{4.66}$$

We have finally to see that the recursion formula (4.66) allows us to choose A_n, B_n such that $A_n B_n$ converges to $K_1(\alpha)$ (see (4.50)). Now, (4.66) is in essence a condition on the *asymptotic* behavior of A_n, B_n as $n \to \infty$. Namely, for small n the inequalities can be easily satisfied by picking A_n large and B_n small (in particular, we need $A_1 B_1$ small because of $g \geqslant h_1$). The asymptotic analysis runs as follows.

First, since g' is regularly varying (with index $\alpha - 1$), we have

$$\frac{e_n}{e_{n+1}} = 1 - \frac{e_{n+1}-e_n}{e_{n+1}} = 1 - \frac{1}{2-\alpha}\frac{1}{n} + o\left(\frac{1}{n}\right) \tag{4.67}$$

(because $e_n g'(e_n)/g(e_n) \sim \alpha$ by [BGT] Theorem 1.5.11). Therefore, using that $\delta_n = o(1/n)$, we have

$$(1-\delta_n)\frac{e_n}{e_{n+1}}\frac{n+1}{n} = 1 - \frac{\alpha-1}{2-\alpha}\frac{1}{n} + o\left(\frac{1}{n}\right). \tag{4.68}$$

It follows that if

$$A_n \downarrow A \geqslant \frac{\alpha - 1}{2 - \alpha} \qquad B_n \uparrow B > 0 \tag{4.69}$$

(with A_n converging slow enough and B_n converging fast enough), then (4.66) will be satisfied. Here we have to choose A, B such that the conditions in Lemma 17 can still be satisfied, in other words

$$b - AB > \sqrt{D} \qquad D = (b - AB)^2 + 4aA > 0 \tag{4.70}$$

with a, b the limits of a_n, b_n as $n \to \infty$.

Both a and b depend on the parameter c in the definition of (l_n), which we can still choose. We have (recall (4.43) and (4.46))

$$a = -(\alpha - 1)\, c^{\alpha}, \qquad b = \alpha c^{\alpha - 1}. \tag{4.71}$$

For fixed a, b, A the maximal value of AB subject to (4.70) is

$$AB = b - \sqrt{4(-a)\, A}. \tag{4.72}$$

(Note that $4aA < 0$, so the second condition in (4.70) implies the first. The value in (4.72) is in fact the boundary case with $D = 0$.) Substitute (4.71) and pick $A = (\alpha - 1)/(2 - \alpha)$ as best choice, to get

$$AB = \alpha c^{\alpha - 1} - \sqrt{4\, \frac{(\alpha - 1)^2}{2 - \alpha}\, c^{\alpha}}. \tag{4.73}$$

Maximize the r.h.s. over c. The maximizer is

$$c_{\max} = (2 - \alpha)^{1/(2 - \alpha)} \tag{4.74}$$

and consequently the maximal value for AB is $c_{\max} = K_1(\alpha)$, as claimed in Lemma 18.

The ε in (4.50) is needed to allow for strict inequality in (4.70) and to ensure that $\liminf_{n \to \infty} e^-_{n+1}/e_n = \liminf_{n \to \infty} e^-_{n+1}/e_{n+1} \geqslant \delta(\varepsilon)$ (see (4.49)). ∎

Proof of Lemma 19. We shall estimate the function Δ_n from above by a function that can be explicitly calculated.

Recall from (4.56) that $f_n \leqslant g_n^{\cup}$, with equality on the interval $[0, e_n^+]$. Moreover, $g_n^{\cup} - f_n$ is convex. Therefore it follows from Proposition 4(g′) that

$$\Delta_n = K_{f_n}(g_n^{\cup} - f_n) \leqslant K_{g_n^{\cup}}(g_n^{\cup} - f_n) \leqslant K_{g_n^{\cup}}(g_n^{\cup} 1_{[e_n^+, \infty)}). \tag{4.75}$$

Now, the kernel $K_{g_n^{\cup}}$ is known explicitly. Indeed, a straightforward computation gives (see (0.2), (1.12) and (4.40))

$$K_{g_n^{\cup}}(\theta, dx) = \frac{dx}{Z_n(\theta)} \frac{1}{(A_n/n)\, x(B_n e_n + x)} \times \left[\left(\frac{x}{\theta}\right)^{\theta/e_n} \left(\frac{B_n e_n + x}{B_n e_n + \theta}\right)^{-(B_n e_n + \theta)/e_n} \right]^{n/A_n B_n}, \tag{4.76}$$

where $Z_n(\theta)$ is the normalizing constant. Next, pass to the scale e_n by putting $x = ze_n$, $\theta = \tau e_n$, and define

$$\begin{aligned} r_n^- &= e_{n+1}^-/e_n, \qquad & s_n^- &= e_n^-/e_n \\ r_n^+ &= e_{n+1}^+/e_n, \qquad & s_n^+ &= e_n^+/e_n. \end{aligned} \tag{4.77}$$

Estimate

$$g_n^{\cup}(ze_n) = \frac{A_n}{n} ze_n(B_n e_n + ze_n) \leqslant \left[A_n \left(\frac{B_n}{s_n^+} + 1 \right) \frac{e_n^2}{n} \right] z^2 \qquad (z \geqslant s_n^+). \tag{4.78}$$

Combine (4.76–4.78) with the bound $g_n^{\cup}(\tau e_n) \geqslant [A_n B_n (e_n^2/n)]\, \tau$ to obtain the estimate

$$\Delta_n(\tau e_n) \leqslant \left[\frac{1}{B_n} \left(\frac{B_n}{s_n^+} + 1 \right) \right] g_n^{\cup}(\tau e_n) \frac{1}{\tau} I_n(\tau) \qquad (\tau > 0) \tag{4.79}$$

with

$$I_n(\tau) = \int_{s_n^+}^{\infty} \frac{dz}{Z_n(\tau)} z^2 \frac{1}{z(B_n + z)} [z^{\tau}(B_n + z)^{-(B_n + \tau)}]^{n/A_n B_n}, \tag{4.80}$$

where $Z_n(\tau)$ is the normalizing constant (i.e., the integral over $[0, \infty)$ but without the term z^2).

To proceed we shall need the following:

LEMMA 20. *Suppose that* (4.69–4.70) *hold. Then there exist* $0 < t^- < t^+ < \infty$ *such that*

$$\begin{aligned} \lim_{n\to\infty} r_n^- &= \lim_{n\to\infty} s_n^- = t^- \\ \lim_{n\to\infty} r_n^+ &= \lim_{n\to\infty} s_n^+ = t^+. \end{aligned} \tag{4.81}$$

Proof. Elementary. See the proof of Lemma 17 and use that $\lim_{n\to\infty} e_n/e_{n+1} = 1$. ∎

To complete the proof of Lemma 19 it now suffices to show that (see (4.64) and (4.79))

$$\sup_{0<\tau\leqslant r_n^-} \frac{1}{\tau} I_n(\tau) = o\left(\frac{1}{n}\right). \tag{4.82}$$

This goes as follows. Estimate

$$\begin{aligned} Z_n(\tau) &\geqslant \int_0^\tau dz \, \frac{1}{z(B_n+z)} \left[z^\tau (B_n+z)^{-(B_n+\tau)} \right]^{n/A_n B_n} \\ &\geqslant \frac{1}{B_n+\tau} \left[(B_n+\tau)^{-(B_n+\tau)} \right]^{n/A_n B_n} \int_0^\tau dz \, z^{\tau(n/A_n B_n)-1} \\ &= \frac{A_n B_n}{\tau n(B_n+\tau)} \left[\tau^\tau (B_n+\tau)^{-(B_n+\tau)} \right]^{n/A_n B_n}. \end{aligned} \tag{4.83}$$

Hence

$$\frac{1}{\tau} I_n(\tau) \leqslant \frac{n(B_n+\tau)}{A_n B_n} \int_{s_n^+}^\infty dz \, \frac{z}{B_n+z} \left[\left(\frac{z}{\tau}\right)^\tau \left(\frac{B_n+z}{B_n+\tau}\right)^{-(B_n+\tau)} \right]^{n/A_n B_n}. \tag{4.84}$$

Now, the term between square brackets in (4.84) uniquely assumes the maximal value 1 at $z = \tau$. Hence it follows from the separation of s_n^+ and r_n^- stated in Lemma 20 that

$$\sup_{0<\tau\leqslant r_n^-} \frac{1}{\tau} I_n(\tau) = O(e^{-\beta n}) \qquad \text{for some} \quad \beta > 0, \tag{4.85}$$

which is much stronger than (4.82). ∎

Step 2. Removal of the restrictions on g.

This is completely analogous to Step 2 in Section 4.2. The details are left to the reader.

4.5. *Upper Bound for* $\alpha \in (0, 1)$

The argument here runs essentially parallel to that of Section 4.4 and complements the lower bound derived in Section 4.2. We replace the up-parabolas by down-parabolas

$$g_n^{\cap}(x) = \frac{A_n}{n}\, x(B_n e_n - x)^{+}. \tag{4.86}$$

The steps in the proof are the same (with all the inequality signs reversed) except for two points: (I) the definition of the function f_n in (4.56) has to be modified; (II) the integral in (4.80) comes out a bit different. We shall only describe these changes, skipping the rest of the argument.

(I) We keep the definition of l_n in (4.41), which now dominates (and touches) g from above. The new definition of f_n replacing (4.56) is

$$f_n = g_n^{\cap} \wedge l_n. \tag{4.87}$$

The definition of h_n is the same as in (4.45) but with $g_n^{\cap}$. First, since $f_n \leqslant h_n$ and h_n is concave, we have by Proposition 4(g′)

$$Fh_n = K_{h_n}(h_n) \leqslant K_{f_n}(h_n) = K_{f_n}(f_n) + K_{f_n}(h_n - f_n). \tag{4.88}$$

Second, $f_n \leqslant g_n^{\cap}$ and hence

$$K_{f_n}(f_n) = Ff_n \leqslant Fg_n^{\cap} = \left(1 + \frac{A_n}{n}\right)^{-1} g_n^{\cap} \tag{4.89}$$

(see Proposition 3(a) and Lemma 8(b)). Third, since $h_n - f_n$ is convex, Proposition 4(g′) also gives

$$K_{f_n}(h_n - f_n) \leqslant K_{g_n^{\cap}}(h_n - f_n) \leqslant l_n(B_n e_n)\, K_{g_n^{\cap}}(1_{[e_n^-,\, B_n e_n]}). \tag{4.90}$$

Combine (4.88–4.90) to get

$$Fh_n \leqslant \left(1 + \frac{A_n}{n}\right)^{-1} (g_n^{\cap} - \Delta_n) \tag{4.91}$$

with

$$\Delta_n = \left(1 + \frac{A_n}{n}\right) l_n(B_n e_n)\, K_{g_n^{\cap}}(1_{[e_n^-,\, B_n e_n]}). \tag{4.92}$$

The integral can, similarly as in (4.83–4.84), be estimated in an elementary way and one arrives at an estimate of the form (4.85).

(II) Equation (4.76) is replaced by

$$K_{\hat{g}_n}(\theta, dx) = \frac{dx}{Z_n(\theta)} \frac{1}{(A_n/n)\, x(B_n e_n - x)} \left[\left(\frac{x}{\theta}\right)^{\theta/e_n} \left(\frac{B_n e_n - x}{B_n e_n - \theta}\right)^{(B_n e_n - \theta)/e_n} \right]^{n/A_n B_n}. \tag{4.93}$$

Correspondingly, the integral in (4.80) now runs only up to $B_n e_n$. The crucial estimate is again (4.82), now with

$$I_n(\tau) = \int_{s_n^+}^{B_n} \frac{dz}{Z_n(\tau)} \frac{1}{z(B_n - z)} \left[z^{\tau} (B_n - z)^{(B_n - \tau)} \right]^{n/A_n B_n}. \tag{4.94}$$

Etcetera.

4.6. *Proof of Theorem* 6(ii–iii)

Theorem 6(i) has already been proved, since the estimates in Sections 4.2–4.5 are valid for $\theta \in (0, o(e_n))$. The proof of Theorems 6(ii–iii) below is a slight variation on the argument in Section 3.2.

We need a small preparatory lemma.

LEMMA 21. *If g satisfies* (4.1), *then*

$$\lim_{\theta \to \infty} \frac{(Fg)(\theta)}{g(\theta)} = 1. \tag{4.95}$$

Proof. Left to the reader. Recall (1.1–1.2). ∎

(ii) $\underline{\liminf_{x \downarrow 0} x^{-2} g(x) > 0.}$ Consider first the case $\alpha \in (0, 1)$. Return to the proof of Lemma 13(a). Equation (3.8) tells us that $F^{n_0+2} g$ has a *strictly positive* slope at 0. Apply the argument of Section 4.2 to $F^{n_0+2} g$ instead of g. The down-parabola can be fitted under $F^{n_0+2} g$ all the way up to 0. Moreover, by Lemmas 14 and 21, $F^{n_0+2} g$ has the same sequence (e_n) as g asymptotically. Hence we get the same lower bound as in Section 4.2 for the scaled functions

$$\frac{1}{\theta} \frac{n}{e_n} (F^{n+n_0+2} g)(\theta). \tag{4.96}$$

This in turn leads to the desired lower bound as $n \to \infty$, because $\lim_{n \to \infty} e_{n+n_0+2}/e_n = 1$.

To get the complementary upper bound, we use the concave upper envelope g^+ of g. Since $F^n g^+$ is concave by Proposition 3(b) and dominates $F^n g$, it suffices to get an upper bound on the slope at 0 of the scaled functions

$$\frac{1}{\theta}\frac{n}{e_n}(F^n g^+)(\theta). \tag{4.97}$$

Now, (3.8) tells us that

$$\lim_{\theta\downarrow 0}\frac{1}{\theta}\left[\frac{n}{e_n}(F^n g^+)(\theta)\right]=\int_0^\infty du\exp\left[-\int_0^u dv\,\frac{1}{h_n(v)}\right] \tag{4.98}$$

with

$$h_n(v)=\frac{1}{v(e_n/n)}\left[\frac{n}{e_n}(F^n g^+)\left(v\,\frac{e_n}{n}\right)\right]. \tag{4.99}$$

Since g^+ has the same sequence (e_n) as g asymptotically, the argument in Section 4.5 provides us with an upper bound on $(n/e_n)\,F^n g^+$. Substitution into (4.98) gives the desired upper bound on the slope at 0.

The same argument works for the case $\alpha\in(1,2)$ and follows the arguments in Sections 4.3–4.4.

(iii) $\underline{\limsup_{x\downarrow 0} x^{-2}g(x)=0.}$ Trivial by Lemma 13(b).

5. PROOF OF THEOREM 7

In this section we briefly explain why Theorem 7 is a straightforward generalization of Theorem 4.

Our heuristic argument in Section 4.1 was based on Proposition 2 in Section 1.3, which describes the action of the kernels $K_g^{(n)}$ $(n\geqslant 1)$ defined in (1.13). For the generalized transformation defined in (0.17–0.18) the new kernels are

$$K_g^{(n)}=K_{F^{(n-1)}g}\circ\cdots\circ K_{F^{(0)}g}\qquad(n\geqslant 1) \tag{5.1}$$

with

$$F^{(n)}=F_{c_{n-1}}\circ\cdots\circ F_{c_0}\qquad(n\geqslant 1). \tag{5.2}$$

We easily deduce that the new version of Proposition 2 reads:

$$\begin{aligned}
&\text{(a)} \quad \int_0^\infty K_g^{(n)}(\theta, dy) = 1 \\
&\text{(b)} \quad \int_0^\infty y K_g^{(n)}(\theta, dy) = \theta \\
&\text{(c)} \quad \int_0^\infty y^2 K_g^{(n)}(\theta, dy) = \theta^2 + \sigma_n (F^{(n)} g)(\theta) \\
&\text{(d)} \quad \int_0^\infty g(y)\, K_g^{(n)}(\theta, dy) = (F^{(n)} g)(\theta)
\end{aligned} \tag{5.3}$$

for all $g \in \mathscr{H}$, $\theta \in [0, \infty)$ and $n \geqslant 1$, with σ_n given by (0.20):

$$\sigma_n = \sum_{k=0}^{n-1} \frac{1}{c_k}. \tag{5.4}$$

The action of $K_g^{(n)}$ on straight lines, up- and down-parabolas becomes

$$\begin{aligned}
K_g^{(n)}(g_{a,b}) &= g_{a,b} && (a > 0,\ b \in \mathbb{R}) \\
K_g^{(n)}(g_{a,b}^{\cup}) &= \frac{1}{1 - a\sigma_n}\, g_{a,b}^{\cup} && \left(0 < a < \frac{1}{\sigma_n},\ b \in \mathbb{R}\right) \\
K_g^{(n)}(g_{a,b,c}^{\cap}) &= \frac{1}{1 + a\sigma_n}\, g_{a,b,c}^{\cap} && (a > 0,\ b,\ c \in \mathbb{R}),
\end{aligned} \tag{5.5}$$

generalizing Lemmas 7–8. This forms the basis for the rigorous bounds that were derived in Sections 4.2–4.5.

Thus, all we have to do is replace (F^n, n) by $(F^{(n)}, \sigma_n)$. All the calculations in Section 4 carry over with the *same constants* everywhere. Only the defining relation for e_n needs to be modified into (0.21), resulting in all free factors n changing into σ_n. The reader will be easily convinced that there is no snag.

A. APPENDIX A

In this appendix we give an outline of the main steps in the proof of Propositions 4(e′–f′). The technicalities will be worked out in Appendix B. Proposition 4(g′) will be proved in Appendix C as a corollary.

A.1. *Informal Background*

Return to the stochastic differential equation in (0.26). In terms of the solution to this equation, $(Z_\theta^g(t))_{t\geqslant 0}$, we can write

$$(Fg)(\theta)=\int_0^\infty g(x)\,\nu_\theta^g(dx)$$
$$=\lim_{t\to\infty} E(g(Z_\theta^g(t))\mid Z_\theta^g(0)=x). \tag{A.1}$$

Namely, $Z_\theta^g(t)$ as $t\to\infty$ converges in law to ν_θ^g, the unique equilibrium of (0.26). The limit in (A.1) is independent of the initial value $Z_\theta^g(0)=x$ as long as $x>0$. This is because the diffusion in (0.26) is ergodic, which can be proved easily by coupling (see [DG2]). The semigroup $(S_\theta^g(t))_{t\geqslant 0}$ on $L^1([0,\infty);\nu_\theta^g)$ associated with (0.26) is

$$(S_\theta^g(t)\,f)(x)=E(f(Z_\theta^g(t))\mid Z_\theta^g(0)=x). \tag{A.2}$$

The generator of this semigroup is the closure of the operator $(\theta-x)(\partial/\partial x)+g(x)(\partial^2/\partial x^2)$. Thus, informally, if we define $u(\theta,x,t)=(S_\theta^g(t)\,f)(x)$ then this function satisfies the equation

$$\begin{aligned} u_t&=(\theta-x)\,u_x+g(x)\,u_{xx} && (x,t>0)\\ u(\theta,x,0)&=f(x) && (x>0). \end{aligned} \tag{A.3}$$

One way to prove (e′) and (f′) would be to show that $u_\theta(\theta,x,t)$ and $u_{\theta\theta}(\theta,x,t)$ have a sign for all $x,t>0$ and that this sign is the same as that of f' resp. f''. Namely, we could then draw out (e′) and (f′) by observing that this property would be inherited by the limit (recall (1.12))

$$(K_g f)(\theta)=\int_0^\infty f(x)\,\nu_\theta^g(dx)=\lim_{t\to\infty} u(\theta,x,t). \tag{A.4}$$

We shall essentially follow this strategy here. However, we use the Laplace transform w.r.t. the time variable as the basic object. This way we avoid t-derivatives.

Define

$$\hat u(\theta,x,\lambda)=\frac{1}{\lambda}\int_0^\infty e^{-(t/\lambda)}u(\theta,x,t)\,dt \qquad (\lambda>0). \tag{A.5}$$

Then (A.4) gives

$$(K_g f)(\theta)=\lim_{\lambda\to\infty}\hat u(\theta,x,\lambda). \tag{A.6}$$

We shall obtain (e′) and (f′) by proving the following result: For $g \in \mathscr{H}$ and $f\colon [0, \infty) \to \mathbb{R}$ such that $f \in \bigcap_{\theta \in [0, \infty)} L^1([0, \infty); \nu_\theta^g)$,

(e″) f increasing (decreasing) $\Rightarrow \theta \to \hat{u}(\theta, x, \lambda)$ increasing (decreasing) for all $x, \lambda > 0$.

(f″) f convex (concave) $\Rightarrow \theta \to \hat{u}(\theta, x, \lambda)$ convex (concave) for all $x, \lambda > 0$.

(Note here that both statements are true for *arbitrary* $g \in \mathscr{H}$. The restriction is on f.)

The way we approach the proof of (e″) and (f″) is via a standard-type *variational argument*. Define, still informally,

$$\begin{aligned} p &= \hat{u} \\ q &= \hat{u}_x. \end{aligned} \tag{A.7}$$

Keep θ, λ fixed and take x as the running variable.

LEMMA 22. *The functions p and q satisfy*:

(a) $p = \lambda[\, g p_{xx} + (\theta - x)\, p_x] + f$.

(b) $(1 + \lambda)\, q = \lambda[\, g q_{xx} + (g' + (\theta - x))\, q_x] + f'$.

Proof. Use (A.3–A.5). ∎

Fix $\theta, \lambda > 0$. We shall use the representations in Lemma 22 to prove the following three positivity principles:

$$\begin{aligned} &(1) \quad f \geqslant 0 \Rightarrow p \geqslant 0 \\ &(2) \quad f' \geqslant 0 \Rightarrow p_x = q \geqslant 0 \\ &(3) \quad f'' \geqslant 0 \Rightarrow p_{xx} = q_x \geqslant 0. \end{aligned} \tag{A.8}$$

A precise formulation of (A.8)(1–3) will be given in Propositions 5–7 below. Here we explain informally how (e″) and (f″) follow.

Proof of (e″). For fixed $\Delta > 0$, put

$$r(x) = p(\theta + \Delta, x, \lambda) - p(\theta, x, \lambda). \tag{A.9}$$

From Lemma 22(a) and $p_x = q$ it follows that

$$r - \lambda[\, g r_{xx} + (\theta - x)\, r_x] = \lambda \Delta\, q(\theta + \Delta, x, \lambda). \tag{A.10}$$

This is the same equation as in Lemma 22(a), but with p replaced by r and f replaced by $x \to \lambda\Delta\, q(\theta+\Delta, x, \lambda)$. Applying (A.8)(1–2), we therefore get

$$f' \geqslant 0 \Rightarrow q \geqslant 0 \Rightarrow r \geqslant 0, \tag{A.11}$$

which is (e″) because of (A.7) and (A.9). ∎

Proof of (f″). Fix $\Delta > 0$. For $0 < \alpha < 1$, define

$$s(x) = (1-\alpha)\, p(\theta, x, \lambda) + \alpha p(\theta+\Delta, x, \lambda) - p(\theta+\alpha\Delta, x, \lambda). \tag{A.12}$$

¿From Lemma 22(a) it follows that

$$\begin{aligned} s - \lambda[\, g s_{xx} + (\theta - x)\, s_x] &\\ = (1-\alpha)\, f + \alpha[\, f + \lambda\Delta\, q(\theta+\Delta, x, \lambda)] - [\, f + \lambda\alpha\Delta\, q(\theta+\alpha\Delta, x, \lambda)] &\\ = \lambda\alpha\Delta\, [\, q(\theta+\Delta, x, \lambda) - q(\theta+\alpha\Delta, x, \lambda)]. & \end{aligned} \tag{A.13}$$

This is the same equation as in Lemma 22(a), but with p replaced by s and f replaced by the last term. Hence, if we could show that

$$f'' \geqslant 0 \Rightarrow q(\theta+\Delta, x, \lambda) - q(\theta+\alpha\Delta, x, \lambda) \geqslant 0, \tag{A.14}$$

then we would obtain $s \geqslant 0$ by (A.8)(1), which is (f″) because of (A.7) and (A.12). To prove (A.14), put $\tilde{\theta} = \theta + \alpha\Delta$ and $\tilde{\Delta} = (1-\alpha)\, \Delta$, and define

$$t(x) = q(\tilde{\theta} + \tilde{\Delta}, x, \lambda) - q(\tilde{\theta}, x, \lambda). \tag{A.15}$$

Substitution into Lemma 22(b) gives

$$(1+\lambda)\, t - \lambda[\, g t_{xx} + (g' + (\tilde{\theta} - x))\, t_x] = \tilde{\Delta}\tilde{\theta}\, q_x(\tilde{\theta} + \tilde{\Delta}, x, \lambda). \tag{A.16}$$

According to (A.8)(2), it now remains to show that

$$f'' \geqslant 0 \Rightarrow q_x(\tilde{\theta} + \tilde{\Delta}, x, \lambda) \geqslant 0. \tag{A.17}$$

But this is precisely (A.8)(3). ∎

The technicalities behind (A.8) and (A.9–A.17) are slightly delicate, because we are dealing with functions on the non-compact set $[0, \infty)$. Therefore we cannot use standard positivity principles from the literature (see e.g. [PW]). In Sections A.2–A.3 we give an outline of the main steps. These will be based on three propositions and three lemmas, the proof of which is deferred to Appendix B.

A.2. *Proof of* (e″)

The strategy of the proof of (A.8)(1–3) is to reformulate (a) and (b) in Lemma 22 as *variational problems*. However, here a number of technical difficulties arise due to the fact that the x-variable runs over an unbounded domain. We shall therefore first discuss the problem not for general $g \in \mathscr{H}$ but for g satisfying some appropriate restrictions. Later we shall remove these restrictions via an approximation argument.

Below we abbreviate $I = [0, \infty]$, and $int(I)$ denotes the interior of I. (All statements below will in fact be proved for arbitrary $I = [A, B]$ with $-\infty \leqslant A < B \leqslant \infty$.)

DEFINITION ("class $\mathscr{R}$"). $\mathscr{R}$ is the set of functions $g\colon \mathbb{R} \to [0, \infty)$ such that

(i) g is locally Lipschitz continuous

(ii) there exist $0 < c \leqslant a < \infty$ and $0 < d \leqslant b < 1$ such that

$$c + dx^2 \leqslant g(x) \leqslant a + bx^2 \qquad (x \in \mathbb{R}). \tag{A.18}$$

The domain of equations (a) and (b) in Lemma 22 has to be specified properly and for this purpose we need to introduce classes $\mathscr{H}_i(g, \alpha, \theta)$ with $i = 0, 1, 2$ and $\alpha = 0, 1, 2$ as follows.

DEFINITION ("classes $\mathscr{H}_i(g, \alpha, \theta)$"). For $g \in \mathscr{R}$ and $\theta \in int(I)$, let

$$\begin{aligned} \mathscr{H}_0(g, \alpha, \theta) &= \{v\colon \mathbb{R} \to \mathbb{R} \text{ measurable: (i) and (ii) hold}\} \\ \mathscr{H}_1(g, \alpha, \theta) &= \{v \in \mathscr{H}_0(g, \alpha, \theta)\text{: (iii) and (iv) hold}\} \\ \mathscr{H}_2(g, \alpha, \theta) &= \{v \in \mathscr{H}_1(g, \alpha, \theta)\text{: (v) and (vi) hold}\} \end{aligned} \tag{A.19}$$

with

$$\begin{aligned} &\text{(i)} \quad v = 0 \text{ on } I^c \\ &\text{(ii)} \quad \int_I dx\, g^\alpha \mu_\theta^g v^2 < \infty \\ &\text{(iii)} \quad v \in AC(int(I)) \\ &\text{(iv)} \quad \int_I dx\, g^{\alpha+1} \mu_\theta^g v_x^2 < \infty \\ &\text{(v)} \quad g^{\alpha+1} \mu_\theta^g v_x \in AC(int(I)) \\ &\text{(vi)} \quad \int_I dx\, [(g^{\alpha+1} \mu_\theta^g v_x)_x]^2 \frac{1}{g^\alpha \mu_\theta^g} < \infty. \end{aligned} \tag{A.20}$$

Note that $\mathscr{H}_0(g, 0, \theta) = L^2(I; \mu_\theta^g)$. With these ingredients we can now prove the following two positivity principles (which give a precise meaning to (A.8)(1–2)):

PROPOSITION 5. *Fix* $\theta \in int(I)$. *Let* $g \in \mathscr{R}$, $\lambda > 0$ *and* $f \in \mathscr{H}_0(g, 0, \theta)$. *Consider the equation*

$$\begin{gathered} p = \lambda[g p_{xx} + (\theta - x)\, p_x] + f \\ p \in \mathscr{H}_1(g, 0, \theta), p_x \in AC(int(I)). \end{gathered} \tag{A.21}$$

(a) *The solutions of* (A.21) *are minimizers of the variational problem*

$$\min_{p \in \mathscr{H}_1(g, 0, \theta)} \int_{\mathbb{R}} dx\, \{ p^2 + \lambda g p_x^2 - 2fp \}\, \mu_\theta^g. \tag{A.22}$$

(b) *This variational problem has a unique minimizer* p^*, *satisfying* $p^* \in \mathscr{H}_2(g, 0, \theta)$, $p_x^* \in \mathscr{H}_1(g, 1, \theta)$ *and solving* (A.21).

(c) *If* $f \geqslant 0$ *then* $p^* \geqslant 0$.

PROPOSITION 6. *Fix* $\theta \in int(I)$. *Let* $g \in \mathscr{R}$, $\lambda > 0$ *and* $f \in \mathscr{H}_1(g, 0, \theta)$. *Let* p^* *be the solution of* (A.21–A.22) *and put* $q^* = p_x^*$.

(a) q^* *solves the equation*

$$\begin{gathered} (1 + \lambda)\, q = \lambda[g q_{xx} + (g' + (\theta - x))\, q_x] + f' \\ q \in \mathscr{H}_1(g, 1, \theta), q_x \in AC(int(I)). \end{gathered} \tag{A.23}$$

(b) q^* *is the unique minimizer of the variational problem*

$$\min_{q \in \mathscr{H}_1(g, 1, \theta)} \int_{\mathbb{R}} \{ (1 + \lambda)\, q^2 + \lambda g q_x^2 - 2f'q \}\, g\mu_\theta^g \tag{A.24}$$

and satisfies $q^* \in \mathscr{H}_2(g, 1, \theta)$.

(c) *If* $f' \geqslant 0$ *then* $q^* \geqslant 0$.

To give a formal proof of (e″) we need two more facts:

LEMMA 23. *Fix* $g \in \mathscr{R}$ *and* $\theta \in int(I)$. *Then* $p^* = p^*(\theta, x, \lambda)$ *satisfies*

$$\lim_{\lambda \to \infty} p^*(\theta, x, \lambda) = (K_g f)(\theta) \qquad \text{for all} \quad x > 0. \tag{A.25}$$

LEMMA 24. *For all* $g \in \mathscr{R}$, $i = 0, 1$ *and* $\alpha = 0, 1$:

(a) $\mathscr{H}_i(g, \alpha, \theta_2) = \mathscr{H}_i(g, \alpha, \theta_1)$ *for all* $\theta_1, \theta_2 \in int(I)$.

(b) $\mathscr{H}_i(g, \alpha + 1, \theta) \subseteq \mathscr{H}_i(g, \alpha, \theta)$ *for all* $\theta \in int(I)$.

The argument in (A.9–A.11) can now be formalized as follows. Abbreviate

$$\mathcal{H}_i(g, \alpha) = \bigcap_{\theta \in int(I)} \mathcal{H}_i(g, \alpha, \theta)$$
$$L^1(g) = \bigcap_{\theta \in int(I)} L^1(I; \mu_\theta^g). \tag{A.26}$$

Let us assume that $g \in \mathcal{R}$, $f \in \mathcal{H}_1(g, 0)$ and $f' \geqslant 0$. Then $x \to q(\theta + \Delta, x, \lambda)$ satisfies

$$q \in \mathcal{H}_1(g, 1, \theta + \Delta) = \mathcal{H}_1(g, 1, \theta) \subset \mathcal{H}_1(g, 0, \theta), \tag{A.27}$$

by (A.21), (A.23) and Lemmas 23–24. Moreover, $x \to r(x)$ satisfies

$$r \in \mathcal{H}_1(g, 0, \theta),\ r_x \in AC(int(I)), \tag{A.28}$$

by (A.21) and Lemma 24(a). We may therefore apply Propositions 5–6 to (A.10) and obtain that $r \geqslant 0$. The conclusion is that (recall (A.6–A.7))

$$g \in \mathcal{R},\ f \in \mathcal{H}_1(g, 0),\ f' \geqslant 0 \Rightarrow K_g f \text{ increasing}. \tag{A.29}$$

The next step is to pass from $f \in \mathcal{H}_1(g, 0)$, $f' \geqslant 0$ to $f \in L^1(g)$, f increasing. This can be done by a standard approximation: pick (f_n) such that $f_n \in \mathcal{H}_1(g, 0)$, $f'_n \geqslant 0$ and $f_n \to f$ in $L^1(g)$. Since

$$(K_g f)(\theta) = \frac{\int_I f(x)\, \mu_\theta^g(x)\, dx}{\int_I \mu_\theta^g(x)\, dx}, \tag{A.30}$$

it follows that $K_g f_n \to K_g f$ pointwise. Thus we now know that

$$g \in \mathcal{R},\ f \in L^1(g),\ f \text{ increasing} \Rightarrow K_g f \text{ increasing}. \tag{A.31}$$

Finally, we have to show that we can pass from $g \in \mathcal{R}$ to $g \in \mathcal{H}$, the class defined in (0.1). This comes out of the following approximation lemma.

LEMMA 25. *For all $g \in \mathcal{H}$:*

(a) *There exists a sequence $(g_n) \subset \mathcal{R}$ such that $g_n \downarrow g$ pointwise ($\downarrow$ means monotone decreasing as a sequence of functions).*

(b) *If $g_n \downarrow g$ pointwise then $K_{g_n} f \to K_g f$ pointwise for all $f \in L^1(g)$.*

This completes the proof of (e″).

A.3. *Proof of* (f″)

The proof comes out of the following positivity principle (which gives a precise meaning to (A.8)(3)):

PROPOSITION 7. *Fix* $\theta \in int(I)$. *Let* $g \in \mathscr{R}$, $\lambda > 0$ *and* $f \in \mathscr{H}_2(g, 0, \theta)$. *Let* p^* *and* $q^* = p^*_x$ *be the solutions of* (A.21–A.22) *resp.* (A.23–A.24). *Define*

$$\xi^* = g^2 \mu^g_\theta p^*_{xx} = g^2 \mu^g_\theta q^*_x \tag{A.32}$$

and the class

$$\mathscr{K}(g, \theta) = \left\{ v\colon \mathbb{R} \to \mathbb{R} \text{ measurable: } v = 0 \text{ on } I^c, \int_I dx\, (v^2 + g v_x^2) \frac{1}{g^2 \mu^g_\theta} < \infty \right\}. \tag{A.33}$$

(a) ξ^* *solves the equation*

$$\begin{gathered}(1+\lambda)\, \xi = \lambda [g \xi_{xx} - (\theta - x)\, \xi_x] + f'' g^2 \mu^g_\theta \\ \xi \in \mathscr{K}(g, \theta),\ \xi_x \in AC(int(I)).\end{gathered} \tag{A.34}$$

(b) ξ^* *is the unique minimizer of the variational problem*

$$\min_{\xi \in \mathscr{K}(g, \theta)} \int_{\mathbb{R}} dx\, \{ (1+\lambda)\, \xi^2 + \lambda g \xi_x^2 - f'' g^2 \mu^g_\theta \xi \} \frac{1}{g^2 \mu^g_\theta}. \tag{A.35}$$

(c) *If* $f'' \geqslant 0$ *then* $\xi^* \geqslant 0$.

This positivity principle gives us the proof of (f″) when $g \in \mathscr{R}$, $f \in \mathscr{H}_2(g, 0, \theta)$ and $f'' \geqslant 0$. The generalization to $g \in \mathscr{H}$, $f \in L^1(g)$ and f convex follows the route taken in the proof of (e″) and we shall not elaborate on it further.

In Appendix B we prove Propositions 5–7 and Lemmas 23–25.

B. APPENDIX B

The proofs will be given for functions on an arbitrary closed interval $I = [A, B]$ with $-\infty \leqslant A < B \leqslant \infty$. By specializing to the case $A = 0$, $B = \infty$ we recover the situation described in Appendix A and the main body of the paper.

B.1. *Definitions*

Let $\mathscr{H}(I)$ denote the class of functions $g\colon \mathbb{R} \to [0, \infty)$ satisfying

$$\begin{aligned} &\text{(i)} \quad g = 0 \text{ on } I^c \\ &\text{(ii)} \quad g^{-1} \in L^1_{loc}(int(I)) \\ &\text{(iii)} \quad g \text{ globally Lipschitz continuous on } I \\ &\text{(iv)} \quad g(x) \leqslant ax^2 + b \ (x \in \mathbb{R}) \text{ for some } 0 \leqslant a < 1, b \geqslant 0. \end{aligned} \tag{B.1}$$

For $g \in \mathscr{H}(I)$ and $x, \theta \in int(I)$, define

$$\pi_\theta^g(x) = \exp\left[-\int_\theta^x \frac{y-\theta}{g(y)}\, dy \right] \tag{B.2}$$

and

$$\begin{aligned} \mu_\theta^g(x) &= \frac{\pi_\theta^g(x)}{g(x)} && \text{if} \quad g(x) > 0 \\ &= 0 && \text{otherwise.} \end{aligned} \tag{B.3}$$

Note that

$$\pi_\theta^g > 0,\ \pi_\theta^g \in AC(int(I)),\ \mu_\theta^g \in L^1_{loc}(int(I)) \tag{B.4}$$

and

$$\pi_\theta^g = g\mu_\theta^g,\ (\pi_\theta^g)_x = (\theta - x)\, \mu_\theta^g,\ \pi_\theta^g \leqslant 1. \tag{B.5}$$

Before we proceed, let us first check that μ_θ^g is integrable for all $g \in \mathscr{H}(I)$ and $\theta \in int(I)$. Indeed, pick $\alpha, \beta \in int(I)$ such that $\alpha < \theta < \beta$. Then, by (B.5),

$$\int_\alpha^\beta \mu_\theta^g(x)\, dx = \int_\alpha^\beta \frac{g(x)\, \mu_\theta^g(x)}{g(x)}\, dx \leqslant \int_\alpha^\beta \frac{1}{g(x)}\, dx < \infty$$

$$\begin{aligned} \int_A^\alpha \mu_\theta^g(x)\, dx &= \int_A^\alpha \frac{(\theta - x)\, \mu_\theta^g(x)}{\theta - x}\, dx \\ &= \left[\frac{\pi_\theta^g(x)}{\theta - x} \right]_A^\alpha - \int_A^\alpha \frac{\pi_\theta^g(x)}{(\theta - x)^2}\, dx \leqslant \frac{1}{\theta - A} + \frac{1}{\theta - \alpha} < \infty \end{aligned} \tag{B.6}$$

and similarly for $[\beta, B]$ in the latter.

B.2. *Proof of Propositions* 5–7

Recall the definitions of the classes $\mathscr{R}$, $\mathscr{H}_i(g,\alpha,\theta)$ and $\mathscr{K}(g,\theta)$ in Appendix A. In the proof we shall need the following technical lemma.

LEMMA 26. *For $g\in\mathscr{R}$ and $\theta\in int(I)$*:

(a) $\alpha=0,1$: $u\in\mathscr{H}_1(g,\alpha,\theta)$, $v\in\mathscr{H}_2(g,\alpha,\theta)\Rightarrow(g^\alpha\pi_\theta^g uv_x)(\pm\infty)=0$.

(b) *Suppose that ξ satisfies* (A.34). *Then* $u\in\mathscr{K}(g,\theta)\Rightarrow((1/\pi_\theta^g)\,u\xi_x)(\pm\infty)=0$.

Proof. (a) First we show that $(g^\alpha\pi_\theta^g uv_x)(\pm\infty)=l$ exists. Then we show that $l=0$.

For $a<b$, write

$$[g^\alpha\pi_\theta^g uv_x]_a^b=\int_a^b u_x(g^\alpha\pi_\theta^g v_x)\,dx+\int_a^b u(g^\alpha\pi_\theta^g v_x)_x\,dx \tag{B.7}$$

and estimate

$$\begin{aligned}|\text{1-st term}|&\leqslant\left(\int_a^b u_x^2 g^\alpha\pi_\theta^g\,dx\right)^{1/2}\left(\int_a^b v_x^2 g^\alpha\pi_\theta^g\,dx\right)^{1/2}\\ |\text{2-nd term}|&\leqslant\left(\int_a^b u^2 g^{\alpha-1}\pi_\theta^g\,dx\right)^{1/2}\left(\int_a^b [(g^\alpha\pi_\theta^g v_x)_x]^2\frac{1}{g^\alpha\pi_\theta^g}\,dx\right)^{1/2}.\end{aligned} \tag{B.8}$$

All four factors tend to 0 as $a,b\to\pm\infty$, because $u\in\mathscr{H}_1(g,\alpha,\theta)$ and $v\in\mathscr{H}_2(g,\alpha,\theta)$ (see (A.19–A.20)). Hence $(g^\alpha\pi_\theta^g uv_x)(x)\to l$ as $x\to\pm\infty$.

Next, for $a<b$, write (recall that $\pi_\theta^g=g\mu_\theta^g$)

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b g^{\alpha+1}\mu_\theta^g uv_x\,dx\right|\\ &\quad\leqslant\left(\int_a^b g^\alpha\mu_\theta^g u^2\,dx\right)^{1/2}\left(\int_a^b g^{\alpha+1}\mu_\theta^g v_x^2\,dx\right)^{1/2}\frac{1}{b-a}\max_{x\in[a,b]} g^{1/2}(x).\end{aligned} \tag{B.9}$$

Let $b\to\infty$ and use that $g\in\mathscr{R}$, to get that there exists $C<\infty$ such that for all a

$$|l|\leqslant C\left(\int_a^\infty g^\alpha\mu_\theta^g u^2\,dx\right)^{1/2}\left(\int_a^\infty g^{\alpha+1}\mu_\theta^g v_x^2\,dx\right)^{1/2}. \tag{B.10}$$

Now let $a\to\infty$ and use that $u\in\mathscr{H}_0(g,\alpha,\theta)$ and $v\in\mathscr{H}_1(g,\alpha,\theta)$, to obtain $l=0$. Similarly when $a,b\to-\infty$.

(b) A similar argument works here, namely

$$\left[\frac{1}{\pi_\theta^g} u\xi_x\right]_a^b = \int_a^b u_x \left(\frac{1}{\pi_\theta^g}\xi_x\right) dx + \int_a^b u\left(\frac{1}{\pi_\theta^g}\xi_x\right)_x dx \tag{B.11}$$

and

$$\begin{aligned} |\text{1-st term}| &\leqslant \left(\int_a^b u_x^2 \frac{1}{\pi_\theta^g} dx\right)^{1/2} \left(\int_a^b \xi_x^2 \frac{1}{\pi_\theta^g} dx\right)^{1/2} \\ |\text{2-nd term}| &\leqslant \left(\int_a^b u^2 \frac{1}{g\pi_\theta^g} dx\right)^{1/2} \left(\int_a^b \left(\frac{\xi_x}{\pi_\theta^g}\right)_x^2 g\pi_\theta^g\, dx\right)^{1/2}. \end{aligned} \tag{B.12}$$

The first three factors tend to 0 as $a, b \to \pm\infty$, because $u, \xi \in \mathscr{K}(g, \theta)$ (see (A.33)). Only the fourth factor needs closer inspection. We bound the integrand with the help of (A.34):

$$\left(\frac{\xi_x}{\pi_\theta^g}\right)_x^2 g\pi_\theta^g = \frac{1}{\lambda^2}\left((1+\lambda)\frac{\xi}{\pi_\theta^g} - f''\right)^2 g\pi_\theta^g \leqslant \frac{2}{\lambda^2}\left((1+\lambda)^2 \frac{\xi^2}{g\pi_\theta^g} + f''^2 g\pi_\theta^g\right). \tag{B.13}$$

The r.h.s. is integrable because $\xi \in \mathscr{K}(g, \theta)$ (see (A.33)) and $f'' \in \mathscr{H}_0(g, 2, \theta)$ (see (A.19–A.20)). The latter inclusion follows from the assumption that $f \in \mathscr{H}_2(g, 0, \theta)$. Therefore also the fourth factor tends to 0 as $a, b \to \pm\infty$. Etcetera. ∎

Proof of Proposition 5. (a) Use (B.5) to rewrite (A.21) as

$$\begin{gathered} p\mu_\theta^g - \lambda(g\mu_\theta^g p_x)_x - f\mu_\theta^g = 0 \\ p \in \mathscr{H}_1(g, 0, \theta),\ p_x \in AC(int(I)). \end{gathered} \tag{B.14}$$

In order to turn this into the variational problem (A.22), we shall need that $p \in \mathscr{H}_2(g, 0, \theta)$. For this we must check (A.20)(v–vi) for $\alpha = 0$. Part (v) follows from $\pi_\theta^g = g\mu_\theta^g \leqslant 1$ and $p_x \in AC(int(I))$. Part (vi) follows from the estimate

$$[(g\mu_\theta^g p_x)_x]^2 \frac{1}{\mu_\theta^g} = \frac{1}{\lambda^2}(p-f)^2 \mu_\theta^g \leqslant \frac{2}{\lambda^2}(p^2+f^2)\mu_\theta^g \tag{B.15}$$

with the r.h.s. integrable because $p \in \mathscr{H}_1(g, 0, \theta) \subset \mathscr{H}_0(g, 0, \theta)$ (recall Lemma 24(b)) and $f \in \mathscr{H}_0(g, 0, \theta)$ (see (A.19–A.20)).

Now multiply (B.14) by $u \in \mathscr{H}_1(g, 0, \theta)$ and integrate over $\mathbb{R}$ to obtain

$$\int_{\mathbb{R}} dx\ \{up + \lambda g u_x p_x - fu\}\ \mu_\theta^g = 0 \qquad \text{for all} \quad u \in \mathscr{H}_1(g, 0, \theta), \tag{B.16}$$

where we use Lemma 26(a) for $\alpha=0$ to get rid of the boundary terms $(g\mu_\theta^g u p_x)(\pm\infty)=0$. This proves that p is a minimizer of (A.22).

(b) The uniqueness of the minimizer follows from the strict convexity of the integrand in (A.22). We already saw that $p\in\mathscr{H}_2(g,0,\theta)$. To prove $p_x\in\mathscr{H}_1(g,1,\theta)$, note that $p\in\mathscr{H}_1(g,0,\theta)$ trivially gives $p_x\in\mathscr{H}_0(g,1,\theta)$. We already have from (B.14) that $p_x\in AC(int(I))$. Thus it remains to check (A.20)(iv) for $\alpha=1$, i.e., $\int_I dx\, g^2\mu_\theta^g p_{xx}^2<\infty$. But, $(g\mu_\theta^g p_x)_x=[gp_{xx}+(\theta-x)\,p_x]\,\mu_\theta^g$, or

$$g^2\mu_\theta^g p_{xx}^2=[(g\mu_\theta^g p_x)_x]^2\frac{1}{\mu_\theta^g}-(\theta-x)^2\mu_\theta^g p_x^2. \tag{B.17}$$

Hence, by (B.15), it suffices to check that $\int_I dx\,(\theta-x)^2\mu_\theta^g p_x^2<\infty$. However, the latter is implied by $p_x\in\mathscr{H}_0(g,1,\theta)$ because $(\theta-x)^2\leqslant Cg(x)$ for some $C<\infty$ (see (A.18)). It now also easily follows that the minimizer of (A.22) solves (B.14).

(c) Suppose that not $p\geqslant 0$. Then, since $f\geqslant 0$, the integrand of (A.22) does not increase when p is replaced by $|p|$. ∎

Proof of Proposition 6. (a) By differentiating (A.21) w.r.t. x and putting $q=p_x$, we get (A.23). Note that $q\in\mathscr{H}_1(g,1,\theta)$ and $q_x\in AC(int(I))$ because $p\in\mathscr{H}_2(g,0,\theta)$.

(b) Use (B.5) to rewrite (A.23) as

$$\begin{gathered}(1+\lambda)\,q\pi_\theta^g-\lambda(g\pi_\theta^g q_x)_x-f'\pi_\theta^g=0\\ q\in\mathscr{H}_1(g,1,\theta),\ q_x\in AC(int(I)).\end{gathered} \tag{B.18}$$

In order to turn this into the variational problem (A.24), we shall need that $q\in\mathscr{H}_2(g,1,\theta)$. For this we must check (A.20)(v–vi) for $\alpha=1$. Part (v) follows from $\pi_\theta^g=g\mu_\theta^g\leqslant 1$, (B.1)(iv) and $q_x\in AC(int(I))$. Part (vi) follows from the estimate

$$[(g\pi_\theta^g q_x)_x]^2\frac{1}{\pi_\theta^g}=\frac{1}{\lambda^2}((1+\lambda)\,q-f')^2\,\pi_\theta^g\leqslant\frac{2}{\lambda^2}((1+\lambda)^2\,q^2+f'^2)\,\pi_\theta^g \tag{B.19}$$

with the r.h.s. integrable because $q, f'\in\mathscr{H}_0(g,1,\theta)$. Note that the latter inclusion for f' follows from the assumption that $f\in\mathscr{H}_1(g,0,\theta)$.

Now multiply (B.18) by $u\in\mathscr{H}_1(g,1,\theta)$ and integrate over $\mathbb{R}$ to obtain

$$\int_{\mathbb{R}} dx\,\{(1+\lambda)\,uq+\lambda g u_x q_x-f'u\}\,\pi_\theta^g=0\qquad\text{for all}\quad u\in\mathscr{H}_1(g,1,\theta), \tag{B.20}$$

where we use Lemma 26(a) for $\alpha = 1$ to get rid of the boundary terms $(g\pi_\theta^g u q_x)(\pm\infty) = 0$. This proves that q is the unique minimizer of (A.24).

(c) Obvious. ∎

Proof of Proposition 7. (a) By putting $\xi = g\pi_\theta^g q_x$ and computing ξ_x, ξ_{xx} with the help of (B.18), one easily checks that ξ satisfies (A.34). Note that $\xi \in \mathscr{K}(g, \theta)$ and $\xi_x \in AC(int(I))$ because $q \in \mathscr{H}_2(g, 1, \theta)$.

(b) In view of the relation

$$\left(\frac{\xi_x}{\pi_\theta^g}\right)_x = \frac{g\xi_{xx} - (\theta - x)\,\xi_x}{g\pi_\theta^g}, \tag{B.21}$$

we may rewrite (A.34) as

$$(1+\lambda)\frac{\xi}{g\pi_\theta^g} - \lambda\left(\frac{\xi_x}{\pi_\theta^g}\right)_x - f'' = 0. \tag{B.22}$$

Multiply by $u \in \mathscr{K}(g, \theta)$ and integrate over $\mathbb{R}$ to get

$$\int_{\mathbb{R}} dx\,\{(1+\lambda)\,u\xi + \lambda g u_x \xi_x - f'' g\pi_\theta^g u\}\,\frac{1}{g\pi_\theta^g} = 0 \qquad \text{for all} \quad u \in \mathscr{K}(g, \theta), \tag{B.23}$$

where we now use Lemma 26(b) to get rid of the boundary terms $(u\xi_x/\pi_\theta^g)(\pm\infty) = 0$. This proves that ξ is the unique minimizer of (A.35).

(c) Obvious. ∎

B.3. *Proof of Lemmas* 23–25

Proof of Lemma 23. Fix $g \in \mathscr{R}$ and $\theta \in int(I)$. Let A be the operator defined by $Ap = gp_{xx} + (\theta - x)\,p_x$. Then (A.21) reads

$$\begin{gathered} p = [Id - \lambda A]^{-1} f \\ p \in \mathscr{H}_1(g, 0, \theta),\ p_x \in AC(int(I)). \end{gathered} \tag{B.24}$$

A is self-adjoint in $L^2(I; \nu_\theta^g)$, satisfies $\langle Ap, p\rangle_{L^2(I;\,\nu_\theta^g)} \leqslant 0$, and has a 1-dimensional null space given by the constant functions. We therefore have, by a standard argument,

$$\lim_{\lambda \to \infty} [Id - \lambda A]^{-1} f = \langle f, e\rangle_{L^2(I;\,\nu_\theta^g)}\, e, \tag{B.25}$$

with e the constant function equal to 1. The limit is the orthogonal projection onto the null space of A. Note that $\langle f, e\rangle_{L^2(I;\,\nu_\theta^g)} = (K_g f)(\theta)$ (see (A.30)). ∎

Proof of Lemma 24. (a) Fix $g \in \mathscr{R}$. We show that $\mu_{\theta_1}^g$ and $\mu_{\theta_1}^g$ are comparable for all $\theta_1, \theta_2 \in int(I)$. Indeed, write (recall (1.1))

$$\frac{\mu_{\theta_1}^g(x)}{\mu_{\theta_2}^g(x)} = \frac{g(x)\,\mu_{\theta_1}^g(x)}{g(x)\,\mu_{\theta_2}^g(x)}$$
$$= \exp\left[-\int_{\theta_1}^{x} \frac{y-\theta_1}{g(y)}\,dy + \int_{\theta_2}^{x} \frac{y-\theta_2}{g(y)}\,dy \right]$$
$$= \exp\left[(\theta_2-\theta_1) \int_{x}^{\theta_1} \frac{dy}{g(y)} + \int_{\theta_1}^{\theta_2} \frac{\theta_2 - y}{g(y)}\,dy \right]. \tag{B.26}$$

Now, the second integral is independent of x, while the first integral is bounded in absolute value by $\int_{\mathbb{R}} (dy/g(y))$. But the latter is finite because $g \in \mathscr{R}$ implies $g(x) \geqslant c + dx^2$ for some $c, d > 0$ (see A.18)).

(b) Straight from the definitions in (A.19–A.20), because $g \in \mathscr{R}$ implies $g \geqslant c$ for some $c > 0$.

Proof of Lemma 25. (a) Fix $g \in \mathscr{H}(I)$ (recall (B.1)). Define $g_n \in \mathscr{H}(I)$ by

$$g_n(x) = \max\left\{ \frac{1}{n}(1+x^2),\, g(x) \right\} \qquad (x \in \mathbb{R},\, n \geqslant 1). \tag{B.27}$$

Then $g_n \in \mathscr{R}$, $g_n \geqslant g$ and $g_n \downarrow g$ pointwise.

(b) The following holds when $g_n \downarrow g$ pointwise in $\mathscr{H}(I)$ and $f \in \bigcap_{\theta \in int(I)} L^1(I; \mu_\theta^g)$:

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x)\,\mu_\theta^{g_n}(x)\,dx = \int_I f(x)\,\mu_\theta^g(x)\,dx + \sum_{x \in \partial I} f(x)\,\frac{\pi_\theta^g(x)}{\theta - x} \tag{B.28}$$

with the convention that the boundary terms at $x \in \partial I = \{A, B\}$ disappear when $A = -\infty$ and/or $B = \infty$. Indeed, if f is continuously differentiable with compact support, then (B.28) is easily read off from the relation

$$\int_a^b f(x)\,\mu_\theta^{g_n}(x)\,dx = \left[f(x)\,\frac{\pi_\theta^{g_n}(x)}{\theta - x} \right]_a^b$$
$$- \int_a^b \pi_\theta^{g_n}(x)\,d\left(\frac{f(x)}{x-\theta}\right) \qquad (-\infty \leqslant a < b \leqslant \infty), \tag{B.29}$$

in combination with the fact that $\lim_{n\to\infty} \pi_\theta^{g_n} = \pi_\theta^g$ pointwise (recall Lemma 4). The extension to $f \in \bigcap_{\theta \in int(I)} L^1(I; \mu_\theta^g)$ follows from a standard approximation argument.

The boundary terms in (B.28) vanish when $g \in \mathcal{H} \subset \mathcal{H}([0, \infty])$ (compare (0.1) with (B.1)), because $\pi_\theta^g(0) = 0$ by Lemma 3. Since $K_{g_n} f$ and $K_g f$ are defined by (A.30), it follows that $K_{g_n} f \to K_g f$ pointwise. ∎

C. APPENDIX C

In this section we prove Proposition 4(g′). The proof is an easy corollary of the positivity principles established in Appendices A–B.

We begin with a small observation, analogous to Lemma 24.

Lemma 27. *For all $g_1, g_2 \in \mathcal{R}$ with $g_1 \leqslant g_2$, $i = 0, 1$ and $\alpha = 0, 1$:*

$$\mathcal{H}_i(g_2, \alpha, \theta) \subset \mathcal{H}_i(g_1, \alpha, \theta) \qquad \textit{for all} \quad \theta \in int(I). \tag{C.1}$$

Proof. Fix $g_1, g_2 \in \mathcal{R}$ with $g_1 \leqslant g_2$. Then from (B.2–B.3) we have that for all $x, \theta \in int(I)$

$$\frac{g_1(x)\, \mu_\theta^{g_1}(x)}{g_2(x)\, \mu_\theta^{g_2}(x)} = \exp\left[-\int_\theta^x \left\{ \frac{1}{g_1(y)} - \frac{1}{g_2(y)} \right\} (y-\theta)\, dy \right] \leqslant 1. \tag{C.2}$$

Moreover, g_1/g_2 is bounded away from 0 and ∞ by (A.18). Now recall (A.19–A.20). ∎

Fix $\theta \in int(I)$ and $\lambda > 0$. Pick $g_1, g_2 \in \mathcal{R}$ with $g_1 \leqslant g_2$ and $f \in \mathcal{H}_2(g_2, 0, \theta)$ with $f'' \geqslant 0$. Return to Proposition 5. By Lemma 24 we have

$$f \in \mathcal{H}_i(g_j, 0, \theta) \qquad (i = 0, 1;\, j = 1, 2). \tag{C.3}$$

Next, let p_1^* and p_2^* be the unique solutions of (A.21) for the pairs (f, g_1) resp. (f, g_2). Define $\Delta = p_2^* - p_1^*$. Then subtraction in (A.21) gives

$$\begin{gathered} \Delta = \lambda[g_1 \Delta_{xx} + (\theta - x)\, \Delta_x] + (g_2 - g_1)(p_2^*)_{xx} \\ \Delta \in \mathcal{H}_1(g_2, 0, \theta),\ \Delta_x \in AC(int(I)). \end{gathered} \tag{C.4}$$

Because $f'' \geqslant 0$, Propositions 5–7 applied to the pair (f, g_2) give that $(p_2^*)_{xx} \geqslant 0$. Together with $g_1 \leqslant g_2$ we therefore have $(g_2 - g_1)(p_2^*)_{xx} \geqslant 0$. Now apply Proposition 5 to (C.4) to get that $\Delta \geqslant 0$. Recalling Lemma 24 and (A.26), we have now proved that

$$g_1, g_2 \in \mathcal{R},\, g_1 \leqslant g_2,\, f \in \mathcal{H}_2(g_2, 0, \theta),\, f'' \geqslant 0 \Rightarrow K_{g_1} f \leqslant K_{g_2} f. \tag{C.5}$$

The generalization to $g_1, g_2 \in \mathscr{H}$, $g_1 \leqslant g_2$, $f \in L^1(g_2)$, f convex follows the route explained at the end of section A.2.

ACKNOWLEDGMENT

The research in this paper is supported by NATO Grant CRG920680.

REFERENCES

[BCGH] J.-B. Baillon, Ph. Clément, A. Greven, and F. den Hollander, On the attracting orbit of a non-linear transformation arising from renormalization of hierarchically interacting diffusions. Part I: The compact case, *Canad. J. Math.* **47** (1995), 3–27.

[BGT] N. H. Bingham, C. M. Goldie, and J. L. Teugels, "Regular Variation," Encyclopedia of Mathematics and Its Applications, Cambridge Univ. Press, Cambridge, 1987.

[BM] P. M. Bleher and P. Májor, The large-scale limit of Dyson's hierarchical vector-valued model at low temperatures, *Ann. Inst. H. Poincaré* **49** (Série Physique Théorique) (1988), 1–85 (Part I); 86–143 (Part II).

[BK] J. Bricmont and A. Kupiainen, Random walks in asymmetric random environments, *Commun. Math. Phys.* **142** (1991), 345–420.

[CGS] J. T. Cox, A. Greven, and T. Shiga, Finite and infinite systems of interacting diffusions (Part I) *Probab. Theory Related Fields* **102** (1995), 165–197; (Part II) *Math. Nachr.*, to appear.

[D] R. Durrett, A new method of proving phase transitions, *in* "Spatial Stochastic Processes" (K. Alexander and J. Watkins, Eds.), pp. 141–170, Birkhäuser, Boston, 1994.

[DG1] D. A. Dawson and A. Greven, Multiple time scale analysis of hierarchically interacting systems, *in* "Stochastic Processes," A Festschrift in Honour of Gopinath Kallianpur (S. Cambanis, J. K. Ghosh, R. L. Karandikar, and P. K. Sen, Eds.), pp. 41–50, Springer, New York, 1993.

[DG2] D. A. Dawson and A. Greven, Multiple time scale analysis of interacting diffusions, *Probab. Theory Related Fields* **95** (1993), 467–508.

[DG3] D. A. Dawson and A. Greven, Hierarchical models of interacting diffusions: Multiple time scale phenomena, phase transition and pattern of cluster-formation, *Probab. Theory Related Fields* **96** (1993), 435–473.

[DG4] D. A. Dawson and A. Greven, Multiple space-time scale analysis for interacting branching models, *Electron. J. Probab.* **1** (1996), paper No. 14, 1–84.

[DGV] D. A. Dawson, A. Greven, and J. Vaillancourt, Equilibria and quasi-equilibria for infinite collections of interacting Fleming-Viot processes, *Trans. Amer. Math.* **347** (1995), 2277–2360.

[DoG] C. Domb and M. S. Green (Eds.), "Phase Transitions and Critical Phenomena," Vol. 6, Academic Press, London, 1976.

[EFS] A. C. D. van Enter, R. Fernandez, and A. D. Sokal, Regularity properties and pathologies of position-space renormalization group transformations: Scope and limitations of Gibbsian theory, *J. Stat. Phys.* **72** (1993), 879–1167.

[G] G. Gauthier, Multilevel systems of bilinear stochastic differential equations, thesis, Carleton University, Ottawa, Canada, 1995.

[dMP] A. de Masi and E. Presutti, "Mathematical Methods for Hydrodynamic Limits," Lecture Notes in Mathematics, Vol. 1501, Springer, Berlin, 1991.

[PW] M. H. Protter and H. F. Weinberger, "Maximum Principles in Differential Equations," Prentice–Hall, Englewood Cliffs, NJ, 1967.

[Sh] T. Shiga, An interacting system in population genetics, *J. Math. Kyoto Univ.* **20** (1980), 213–243 (Part I); 723–733 (Part II).

[Sp] H. Spohn, "Large Scale Dynamics of Interacting Particle Systems," Springer, Berlin, 1991.

Printed in Belgium